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ABSTRACT

This is part two of a two-part SMSG Programed Algebra Text for high school students. The general plan of the course is to build upon the student's experience with arithmetic. This part begins with factorization of positive integers and then develops the manipulative skills of fractions, exponents, radicals, and polynomials. The text then moves to more advanced topics including rational expressions, equivalent equations, and inequalities. Chapter topics include: factors and divisibility; fractions; exponents; radicals; polynomials and factoring; quadratic polynomials; dividing polynomials; rational expressions; truth sets of open sentences; the graph of a linear equation; graphs of other open sentences in two variables; systems of equations and inequalities; graphs of quadratic polynomials; and functions. Response sheets are contained in the separate "Student's Response Booklet." (MP)

Programed First Course in Algebra

Revised Form H

Student's Text, Part II

Prepared under the supervision of the
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SUGGESTED REFERENCES

The following list of references is offered to you with additional reading and study in some of the topics mentioned in the text. The list draws on the volumes of the NEW MATHEMATICAL LIBRARY (NML), which is a series of monographs written for secondary school students and published by the School Mathematics Study Group. The references are by no means complete, but we hope that they will be helpful in directing your attention to available literature at the appropriate level. In the teachers' commentary is an additional list of references. These cover topics which are not necessarily included in this course but which may be of interest to you.

VOLUMES OF THE NEW MATHEMATICAL LIBRARY TO DATE:

Volume	Author and Title
1	Niven, Ivan NUMBERS: RATIONAL AND IRRATIONAL
2	Sawyer, W. W. WHAT IS CALCULUS ABOUT?
3	Beckenbach, E. and R. Bellman AN INTRODUCTION TO INEQUALITIES
4	Kazarinoff, N. D. GEOMETRIC INEQUALITIES
5	Salkind, C. T. THE CONTEST PROBLEM BOOK
6	Davis, P. J. THE LORE OF LARGE NUMBERS
7	Zippin, Leo USES OF INFINITY
8	Yaglom, I. M. GEOMETRIC TRANSFORMATIONS
9	Olds, C. D. CONTINUED FRACTIONS
10	Ore, Oystein GRAPHS AND THEIR USES

11 HUNGARIAN PROBLEM BOOK I

12 HUNGARIAN PROBLEM BOOK II

13 Aaboe, Asger
EPISODES FROM THE EARLY HISTORY OF MATHEMATICS

14 Grossman, I. and W. Magnus
GROUPS AND THEIR GRAPHS

Following each topic listed below is a set of number pairs such as (1 , 3). The first numeral refers to the volume in the series, and the second, in most cases, to the chapter in that volume. Thus, (1 , 3) refers to Volume 1, Chapter 3. In the case of Volume 6, which is divided by section instead of by chapter, the second numeral refers to the section specified. Volumes 11 and 12 are collections of problems of the Eötvös Competitions for the years 1894 through 1928. These are printed in chronological order. For these, the reference (11 , 1899/3), for example, indicates Volume 11, Problem 3 of the 1899 competition. Those references which we consider to be challenging have an asterisk to the left of the reference designation.

Topic	References
Approximations to $\sqrt{2}$	(7 , 2)
Composite numbers	(6 , 20)
Decimal representation	(1 , 2), (1 , 3), (12 , 1907/3), (12 , 1917/2), (12 , 1925/2), (12 , 1927/2)
Divisibility	*(11 , 1899/3), (12 , 1925/1), (6 , 20)
Equivalence relation	*(10 , 7)
Identity element	(1 , 1)
Inequalities	(1 , 3), (3 , 1), (3 , 2), (3 , 3)
Graphs of inequalities	(3 , 3)
Linear equations	(6 , 22)
Natural numbers and integers	(1 , 1), (12 , 1906/3), *(12 , 1913/1), (12 , 1926/2)
Periodic decimals	(1 , 2)
π	(6 , 19)
Primes	(1 , 1), (1 , Append A), (6 , 20), (13 , 2)
Proofs	(1 , 2)

Properties of numbers	
Associative property	(1 , 1)
Closure	(1 , 1), (1 , 2), (1 , 4)
Commutative property	(1 , 1)
Distributive property	(1 , 1)
Quadratic Equations	*(11 , 1336/2), *(11 , 1339/2)
Real numbers	(1 , 5)
Irrational numbers	(1 , 3), *(1 , 4)
Rational numbers	(1 , 3), *(1 , 4)
Real numbers	(1 , 5)
Unique factorization	(1 , Append B)

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Chapter

Chapter 12

FACTORS AND DIVISIBILITY

12-1. Factors

> In his will a farmer left 11 cows to his 3 sons. The will provided that $\frac{1}{2}$ of the 11 cows should go to son Phil, $\frac{1}{4}$ of the 11 cows to son Dave, and $\frac{1}{6}$ of the cows to Bill. The sons argued about this, because none of them wanted just a piece of a cow, as the will seemed to require. As they were arguing, a stranger came along, leading a cow to market. The stranger heard the boys' plight and said, "That's simple. Include my cow with yours and try again." The boys were delighted, for they now had 12 cows instead of 11. Phil took one-half of these, or 6 cows. Dave took one-fourth of them; that is, 3 cows. Bill took one-sixth of them, or 2 cows. The 11 cows which the farmer had willed were now happily divided. The stranger took his own cow and went on his way.

The outcome of this story may cause you to think that each boy did not get his fair share. But notice that each boy received more than the will provided since

$$6 > \frac{11}{2}, \quad 3 > \frac{11}{4}, \quad \text{and} \quad 2 > \frac{11}{6}.$$

Even so, there must be something "fishy" about the story.

1 Add: $\frac{1}{2} + \frac{1}{4} + \frac{1}{6}$. The sum is

[A] less than 1.

[B] equal to 1.

[C] greater than 1.

The sum is $\frac{11}{12}$. [A] is correct. Thus, the farmer's instructions for distributing his property still left $\frac{1}{12}$ to be distributed.

Regardless of the farmer's arithmetic, this anecdote illustrates one point. In this problem the number 12 was easier to deal with than was the number 11. The reason is that when we divide 12 by 2, or by 4, or by 6, we obtain, in each case, a result which is an integer. The same does not occur when, for example, we divide 11 by 2, 4, or 6.

In the discussion that follows in this section we shall confine our attention to the domain of positive integers. We are going to be particularly interested in products of positive integers. 12 is the product of 6 and a positive integer, since $6 \times 2 = 12$. We shall say, therefore, that 6 is a factor of 12; likewise, 2 is a factor of 12. Similarly, 4 is a factor of 12, since $4 \times 3 = 12$.

2	Is there a positive integer such that 3 times that integer equals 12 ? ______ (yes, no)	yes
3	3 · 4 = 12. Therefore, 3 is a ______ of 12.	factor
4	Is there a positive integer such that 5 times that integer equals 12 ? ______	no
5	5 ______ a factor of 12. (is, is not)	is not
6	If A = {5,6,7}, which element of A is a factor of 12 ? ______	6
7	If B = {2,4,6,8}, which element of B is not a factor of 12 ? ______	8

	Observe that 1 · 12 = 12, so 1 and 12 are also factors of 12. In fact, every positive integer has at least two factors.	
8	One of these is the integer itself and the other one is ______.	1
9	Thus, 11 does have two factors; they are the integers ______ and ______.	11 and 1
	The factors of a positive integer, excluding the integer itself and 1, are called proper factors of that integer.	
10	Thus, 1, 2, 3, 4, 6, 12 are all factors of 12. Of these, 2, 3, 4, and 6 are ______ factors of 12.	proper
11	The integer 11 has two factors, 11 and 1, but it has no ______ factors.	proper
12	Which one of the numbers 2, 3, and 4 has a proper factor? ______	4
13	Which one of 6, 7 and 8 does not have any proper factors? ______	7

Now we are ready for a more precise definition of factor.

The positive integer m is a factor of the positive integer n if there is a positive integer q such that $mq = n$. If m does not equal 1 or n, we say that m is a proper factor of n.

4 is a factor of 8, because $4 \cdot 2 = 8$ and 2 is a positive integer.

$$\begin{array}{ccccc} m & \cdot & q & = & n \\ \downarrow & & \downarrow & & \downarrow \\ 4 & \cdot & 2 & = & 8 \end{array}$$

Notice that if m is a factor of n, then $\frac{n}{m} = q$ is a positive integer.

4 is a factor of 8. $\frac{8}{4}$ is a positive integer, 2.

14 Is $\frac{8}{4}$ a factor of 8 ? ______ (yes,no) — yes

5 is a factor of 15. Is $\frac{15}{5}$ (that is, 3) a factor

15 of 15 ? ______ (yes,no) — yes

16 In general: If ______ is a factor of n, then $\frac{n}{m}$ is also a factor of n. — m

Thus, since 6 is a factor of 24, another factor of

17 24 is $\frac{\square}{\square}$ or 4. — $\frac{24}{6}$

Since 8 is a factor of 40, another factor of 40

18 is $\frac{\square}{\square}$; that is, 5. — $\frac{40}{8}$

Since 5 is a factor of 15, another factor of 15

19 is $\frac{\square}{\square}$ = ______. — $\frac{15}{5}$, 3

To express the fact that 5 is a factor of 15, we sometimes say that 5 <u>divides</u> 15.

Similarly, to indicate that 6 is a factor of 24, we

20 may say that 6 ______ 24. — divides

In general: if m and n are <u>positive</u> integers and

21 if m is a factor of n, we say that m divides ______. — n

22 Thus, we say that 8 ______ 40. — divides

23 We also say 5 ______ 40. — divides

24 Since 20 · ______ = 40, 20 divides 40. — 20 · 2

25 Are there other numbers which divide 40 ? ______ — yes

Alternatively, when m divides n, we sometimes say that n <u>is divisible by</u> m.

26 Thus, we say that 24 ______ ______ ______ 6. — is divisible by

27 Similarly, 40 ______ ______ ______ 8. — is divisible by

If m and n are positive integers and if m is a factor of n, we say that m <u>divides</u> n, or n is <u>divisible</u> by m.

We have been limiting our discussion to positive integers. However, it is sometimes convenient to use the vocabulary of "factors" and "divisibility" in speaking of 0 as well as the positive integers.

28	If a is any positive integer, a( ) = 0.	a(0) = 0
29	Since a(0) = 0, for all positive integers a, we shall say that "a ______ (is, is not) a factor of 0".	is
30	Thus, every positive integer is a factor of ______.	0
31	8(0) = ______, hence	0
32	______ is a factor of 0.	8
33	Since 8 is a factor of 0, we shall say that ______ divides ______.	8, 0
34	Since every positive integer, a, is a factor of 0, we shall say that a ______ 0.	divides

In summary, we shall say that 0 is divisible by every positive integer, but 0 does <u>not</u> divide any number. <u>Division by 0 is not defined</u>.

35	Is 2 a factor of 24 ? ______ (Yes, No), since	Yes
36	(2)( ) = 24.	(2)(12) = 24
37	Is 3 a factor of 24 ? ______	yes
38	Is 4 a factor of 24 ? ______	yes
39	Is 5 a factor of 24 ? ______	no
40	5 is not a factor of 24 since there is no positive integer q such that ( )(q) = 24.	(5)(q) = 24
41	The set of all factors of 24 is { , , , , , , , }.	{1,2,3,4,6,8,12,24}

42	Divide 89 by 13. Is the result an integer? ______	no
43	Is 13 a factor of 89 ? ______	no
44	Does 13 divide 89 ? ______	no
45	Divide 91 by 13. 13 × ______ = 91.	13 × 7 = 91
46	13 ______ (is, is not) a factor of 91.	is

47	91 ______ ______ by 13.	is divisible
48	13 ____________ 91. (divides, does not divide)	divides
49	Is 2 a factor of 86 ? ______	yes
50	Is 3 a factor of 2,179 ? ______	no
51	Is 4 a factor of 10,136 ? ______	yes
52	Is 5 a factor of 8,965 ? ______	yes
53	Is 6 a factor of 133,704 ? ______	yes

If an integer has proper factors, we shall call it factorable. Thus, 12 is factorable. Since 11 has no proper factors, 11 is not factorable.

54 Which of the following lists consists entirely of integers which are not factorable?

[A] 85, 29, 93, 94 [C] 51, 29, 61
[B] 51, 29, 94, 61 [D] 29, 37, 61

Since $85 = 5(17)$, $51 = 3(17)$, $93 = 3(31)$, and $94 = 2(47)$, [A], [B], and [C] contain integers which are factorable. On the other hand, there are no proper factors of 29, 37, or 61. Thus, [D] is correct.

12-2. Tests for Divisibility

There are many occasions when we wish to determine whether a given positive integer is or is not factorable. Unfortunately, it is not always easy to do so. For example, 74,329 is factorable, since it is divisible by 311, but we cannot quickly see that $74{,}329 = 311 \times 239$. On the other hand, our experience with arithmetic enables us to tell at a glance that 10 is a factor of 319,440.

You probably already know tests for divisibility by 2 and by 5, as well as by 10. We shall state these tests, and we shall develop tests for divisibility by 3, by 4, by 6, and by 9.

All of these tests depend on the fact that we ordinarily write a positive integer in decimal notation. That is, 3286 means $3(1000) + 2(100) + 8(10) + 6$. We refer to 6 as the last digit (or the units digit) of 3286. Moreover, every positive integer n may also be written in the form

$$n = 10a + b$$

where a and b are integers and $0 \leq a$, $0 \leq b \leq 9$.

For example:

$$3286 = (10)(328) + 6,$$

and $$765 = (10)(76) + 5.$$

In what follows in this section, we shall think of all numbers under discussion as written in decimal notation. We shall thus be able to use such phrases as "the last digit of the integer", meaning the last digit if the number is written in decimal notation.

	You have already learned, from experience, that an	
1	integer is divisible by 2 (has 2 as a ______)	factor
	if the last digit of the integer is 0, 2, 4, 6,	
2	or ______.	8
3	Otherwise, the integer is not divisible by ______.	2
	For example, the only element of the set	
	{800, 1215, 1492, 1776} that does not have 2 as a	
4	factor is ______.	1215

Although we are familiar with this "test" for divisibility by 2, it is interesting to prove that the test is correct. Our proof offers a pattern for developing other "tests".

	Starting with a given integer n, we may write	
5	$n = 10a + b \quad (0 \leq a,\ \underline{0 \leq b \leq \quad})$	$0 \leq b \leq 9$
	We have seen in Section 12-1 that in order to determine whether 2 is a factor of n we need to determine whether $\frac{n}{2}$ is an integer.	
	We see that $\frac{n}{2} = \frac{10a + b}{2}$	
6	$= \underline{5a + \frac{\square}{\square}}$.	$5a + \frac{b}{2}$
7	n is divisible by 2 if $\frac{n}{2}$ is an ______.	integer
	$\frac{n}{2}$ is an integer if $\frac{b}{2}$ is an integer:	

	There are ten possible values of b. We shall separate these possibilities into two cases.	
	Case 1. If b is 0, 2, 4, 6, or 8,	
8	then $\frac{b}{2}$ is 0, 1, _, _, or 4, and thus $\frac{b}{2}$ is an integer.	0,1,2,3 or 4
	Now: 5 is an integer, a is an integer and $\frac{b}{2}$ is an integer.	
	Hence, $5a + \frac{b}{2}$ is an integer, because of the closure property of the set of integers under addition.	
9	But, from Item 6, $5a + \frac{b}{2} = \frac{n}{2}$. Hence, $\frac{n}{2}$ is an ______.	integer
	It follows from the fact that $\frac{b}{2}$ is an integer that	
10	2 is a ______ of n.	factor
	Case 2. If b is 1, 3, 5, 7, or 9, then $\frac{b}{2}$ is	
11	not an ______.	integer
12	In fact, $\frac{b}{2}$ is $\frac{1}{2}$, $\frac{3}{2}$, _, _, or _.	$\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$ or $\frac{9}{2}$
	Since 5a is an integer and $\frac{b}{2}$ is not an integer, we see that $\frac{n}{2}$, which equals $5a + \frac{b}{2}$,	
13	______ (is, is not) an integer.	is not
	In conclusion, since $\frac{b}{2}$ is not an integer, 2 is	
14	not a ______ of n.	factor
	Therefore, we have proved that our test for divisibility by 2 is correct.	

We shall not attempt such a detailed proof for the other tests for divisibility as we discover them. We shall merely indicate for each how the proof might be carried out.

	You know how to recognize whether an integer has 5 as a factor.	
15	An integer divisible by 5 has either 0 or ______ as its last digit.	5
	Thus, 25, 6070, 1115, and 42810 are all divisible	
16	by ______.	5

To determine whether n is divisible by 5, we examine the last digit of n. If this digit is
17 ______ or ______ we know that 5 is a factor. 0, 5
18 Otherwise, 5 is not a ______. factor

* To prove this test for divisibility by 5, we can follow the pattern of the proof for the test for divisibility by 2.

$n = 10a + b$, a and b integers, $0 \leq a$, and
*19 ______ $\leq b \leq$ ______. $0 \leq b \leq 9$

Whether 5 is a factor of n depends upon whether
*20 $\frac{b}{5}$ is an ______. integer

$$\frac{n}{5} = \frac{10a + b}{5}$$

*21 $$= 2a + \frac{\square}{\square}$$ $2a + \frac{b}{5}$

You should be able to complete the proof. What value(s)
*22 must b have in order to make $\frac{b}{5}$ an integer? ______ 0, 5

If an integer n is divisible by 10, then

$$n = 10q$$

where q is an integer.

23 $$\frac{n}{2} = \frac{10q}{2} = \_\_\_\_\_\_$$ 5q

Since $\frac{n}{2}$ is an integer (namely, 5q) n is divisible by 2.

24 Likewise, $\frac{n}{5} = \frac{10q}{\square} =$ ______, and 2q is an integer. $\frac{10q}{5}$, 2q

25 From this we conclude that n is divisible by ______. 5

We have found: If a number is divisible by 10, then
26 it is also divisible by 2 and ______. 5

It happens to be true also that <u>if</u> a number is divisible by 2 and by 5 <u>then</u> it is divisible by 10. We shall see why in Section 12-3.

Let us take this fact for granted for the moment. Then, since we have tests for divisibility by 2 and by 5, we have a "ready-made" test for divisibility by 10. For n to be divisible by 10 it must be divisible by <u>both</u> 2 and 5.

Consider $n = 10a + b$, a and b integers, $0 \le b \le 9$. For n to be divisible by 2, b must be an element of

27 $P = \{0, 2, \_, \_, \_\}$. — $\{0,2,4,6,8\}$

For n to be divisible by 5, b must be an element of

28 $Q = \{\_, \_\}$. — $\{0,5\}$

For n to be divisible by 10, b must be an element both of set P and of set Q. Thus, b must be an

29 element of $P \cap Q$. $P \cap Q = \{\_\}$. — $\{0\}$

Therefore, n is divisible by 10 if and only if the

30 last digit of n is ______. — 0

Let us try to discover a test for divisibility by 4.

Divide each of: 28, 128, 528, 1028, and 234,528 by 4.

31 Each of these ________ (is, is not) divisible by 4. — is

Consider: 16, 216, 916, 3816, and 10,016.

32 Each of these ________ (is, is not) divisible by 4. — is

33 4 ________ (is, is not) a factor of: 13, 118, 518, 2818, or 10,018. — is not

Examine the numbers above, some of which were divisible by 4 and some which were not divisible by 4. Try to state a rule for divisibility by 4.

It appears that divisibility by 4 is connected with the divisibility by 4 of the number formed by the last two digits.

A positive integer n is divisible by 4 if and only if the number represented by the last two digits of the integer is itself divisible by 4.

For a proof of this rule, see Items *35-*38.

34 Which of the following three sets consists entirely of integers which are divisible by 4?

[A] {8, 316, 412, 538}
[B] {4, 606, 320, 1000}
[C] {12, 324, 692, 1016}

538 and 606 are both divisible by 2, but <u>not</u> by 4. (We know this since neither 38 nor 6 is divisible by 4.) [C] is the correct choice.

*35 $3286 = 3200 + \_\_\_\_\_$ — 86

$= (100)(32) + 86$

We have written 3286 as the sum of a multiple of 100, (100)(32), and a positive integer, 86, which is less than 100.

In fact, any integer n may be written as

$n = 100a + b$, where a and b are integers

*36 and $0 \le a$, $0 \le b \le \_\_\_\_$. — $0 \le b \le 99$

To decide whether n is divisible by 4 we must

*37 decide whether $\frac{n}{4}$ is an _____. — integer

$$\frac{n}{4} = \frac{100a + b}{4}$$

*38 $$= 25a + \frac{\square}{\square}$$ — $25a + \frac{b}{4}$

$\frac{b}{4}$ is an integer provided b is one of the numbers 0, 4, 8, 12, 16, ..., 96. You might complete the proof by yourself.

It is possible to develop a simple rule (test) for divisibility by 3. By careful examination of the following example, you can discover this rule. The test makes use of the fact that $10 = 9 + 1$, $100 = 99 + 1$, $1000 = 999 + 1$, etc., and that each of the numbers 9, 99, 999, etc., is divisible by 3.

39 $537 = 5(100) + 3(10) + \underline{\qquad}$ — 7

40 $= 5(99 + 1) + \underline{3(\quad + 1)} + 7$ — $(9 + 1)$

$= 5(99) + 5(1) + 3(9) + 3(1) + 7$

$= (5 \cdot 11)(9) + 3(9) + 5 + 3 + 7$

$= \big((5 \cdot 11)(9) + 3(9)\big) + 5 + 3 + 7$

41 $= (5 \cdot 11 + 3)9 + \underline{(\quad + \quad + \quad)}$ — $(5 + 3 + 7)$

Clearly, $(5 \cdot 11 + 3)9$ is divisible by 3, since 9 is divisible by 3.

To determine whether 537 is divisible by 3, we need

42 to decide whether $5 + 3 + 7$ is divisible by ______. — 3

$5 + 3 + 7$ looks familiar. We started with the

43 integer ______. — 537

$5 + 3 + 7$ is the sum of the digits of the original integer.

$5 + 3 + 7 = 15$

44 $5 + 3 + 7$ ______ (is, is not) divisible by 3. Hence, — is

45 537 ______ (is, is not) divisible by 3. — is

46 $1237 = 1(1000) + 2(100) + 3(10) + \underline{\qquad}$ — 7

47 $= 1(999 + 1) + 2(99 + 1) + 3(\underline{\qquad}) + 7$ — $9 + 1$

$= 1(111 \cdot 9 + 1) + 2(11 \cdot 9 + 1) + 3(9 + 1) + 7$

$= (1 \cdot 111 \cdot 9 + 2 \cdot 11 \cdot 9 + 3 \cdot 9) + 1 + 2 + 3 + 7$

48 $= (1 \cdot 111 + 2 \cdot 11 + 3)9 + (\underline{1 + 2 + \quad + \quad})$ — $1 + 2 + 3 + 7$

49 $1 + 2 + 3 + 7 = \underline{\qquad}$. 13 is not divisible by 3, — 13

50 hence, $1 + 2 + 3 + 7$ ______ (does, does not) have 3 as a factor. — does not

51 Therefore, 1237 ______ (is, is not) divisible by 3. — is not

20

The general rule is:

> An integer n is divisible by 3 if the sum of the digits of n, written in decimal notation, is divisible by 3.

Do you recognize (from studying the examples) that we also have a rule for divisibility by 9 ?

An integer n is divisible by 9 if the sum of the digits of n, written in decimal notation, is divisible by 9. 3867 is not divisible by 9 since $3 + 8 + 6 + 7 = 24$ and 24 is not divisible by 9.

Which of the following are divisible by 3 ? by 9 ?

	n	Divisible by 3	Divisible by 9	
	105	Yes	No	
52	342	______	______	yes, yes
53	1419	______	______	yes, no
54	42739	______	______	no, no

We shall see later, in Section 12-3, that if a number is divisible by 2 and 3, then it is also divisible by 6. Thus, having a rule for divisibility by 2 and another rule for divisibility by 3, we may state the following rule for divisibility by 6.

> n is divisible by 6 if the last digit of n is 0, 2, 4, 6, or 8 <u>and</u> if the sum of the digits of n is divisible by 3.

Which of the following are divisible by 2 ? by 3 ? by 6 ?

	n	Divisible by 2	Divisible by 3	Divisible by 6	
	729	no	yes	no	
55	1432	______	______	______	yes, yes, yes
56	4004	______	______	______	yes, no, no
57	3111	______	______	______	no, yes, no
58	5111	______	______	______	no, no, no
59	123,456	______	______	______	yes, yes, yes

	Apply the tests given in this section in order to answer the following questions without actually performing the division.	
60	Is 3 a factor of 101,001 ?	Yes [1+1+1=3]
61	Is 3 a factor of 37,199 ?	No[3+7+1+9+9=29]
62	Is 6 a factor of 151,821 ?	No [not "even"]

We have presented tests (or rules) for divisibility by 2, 3, 4, 5, 6, and 9. Why not similar tests for 7 and for 8 ?

Actually, to design a test for divisibility by 8 is not too difficult. If you study the proof for the test for 4 in Items *35-*38, you may get the hint for a proof for divisibility by 8. Briefly, an integer is divisible by 8 if the number represented by the last three digits of the integer is divisible by 8. Note that it is <u>not</u> sufficient for a number to be divisible by 2 and by 4 to be divisible by 8. 1124 is divisible by 2 and 4; yet it is not divisible by 8.

For divisibility by 7 the problem is different--there are tests, but there are no <u>simple</u> tests. If you find the work of this chapter especially interesting, and would like to investigate some of the topics more thoroughly, you will find it worthwhile to do some reading in Number Theory.

On page 29 of the SMSG <u>Study Guides in Mathematics</u> is a bibliography which names several books on this topic, some of which may be in your school library. Also, there is a reference to divisibility in the SUGGESTED REFERENCES at the front of this book.

We conclude this section by developing one further result which is useful later.

	From your multiplication facts you know that,	
63	the product of two even integers is ______, and	even
64	the product of two odd integers is ______.	odd
	In particular,	
65	the square of an ______ integer is even, and	even
66	the square of an ______ integer is odd.	odd

The question we ask is: If we know that the square of a certain integer is even, may we conclude that the integer is even? That is, if we assert that x^2 is even, does it follow that x is even?

The answer to the question is "yes". Let us examine the reasoning we might use to justify this answer.

67	Either x is even or x is ______.	odd
68	We wish to prove that x is ______.	even
	Let us "assume" that x is odd and see where the assumption leads.	
69	<u>If</u> x is odd, then x^2 is ______.	odd
70	But this <u>contradicts</u> our original assertion that x^2 ______,	x^2 is even
71	therefore, x ______ (is, is not) odd and, hence,	is not
72	x is ______.	even

The method of reasoning used in Items 67-72 is of great importance in mathematics. This type of argument is called an <u>indirect</u> argument or <u>proof by contradiction</u>. To prove that a certain statement is true, we "assume" that it is false and then show that this assumption leads to a contradiction. We will see other indirect proofs throughout the remainder of the course.

12-3. Prime Numbers and Prime Factorization

We have been talking about factors of positive integers which are themselves positive integers. That is, when we write $mq = n$ we have been restricting m, q, and n to the domain of positive integers.

	If we start with the positive integer 15, we refer	
1	to 3 and 5 as proper ______ of 15.	factors
2	The factors 3 and 5 are both positive ______.	integers.
	3 and 5 are the only positive integers which are	
3	proper factors of ______.	15
	When we write $15 = 3 \cdot 5$ we say we have factored 15	
4	into proper factors <u>over</u> the set of positive ______.	integers
	Factor 77 into proper factors over the positive integers.	
5	77 = ______.	$7 \cdot 11$

	We could, of course, find other factors of a positive integer if we allowed negat integers as factors.	
6	Thus, $15 = (-3)(\_\_\_\_)$.	-5
7	3, ______, ______, and -5 are all integers which are proper factors of 15 if we allow negative integers as factors.	-3, 5
8	If m is a positive integer and if m is a factor of the positive integer n, then -m is also a ______ of n.	factor

Suppose we are asked to list all the integers which are factors of a given positive integer. Our list would contain the positive factors and their opposites. Thus, if we permit negative integers as factors, we really don't discover any essentially "new" factors.

9	The set of integers which are factors of 6 is ______.	{1, -1, 2, -2, 3, -3, 6, -6}
	This set consists of the positive factors of 6,	
10	namely, ______, ______, ______, ______	1, 2, 3, 6
11	and the ______ of these.	opposites

When we limit ourselves to positive integer factors we say we are factoring over the set of positive integers.

What if we allow the set of rational numbers as the domain in which we look for factors of a given positive integer? In other words, what if we factor over the rational numbers? Let us try an example: What rational numbers would be factors of 15?

	When we think about factoring 15 over the rational numbers (using rational numbers as factors) we see many possibilities.	
12	In this sense $\frac{2}{7}$ is a ______ of 15, since	factor
13	$(\frac{2}{7})(\frac{105}{2}) = \_\_\_\_\_\_$.	15
14	$(-\frac{17}{3})(-\frac{45}{17}) = \_\_\_\_\_\_$.	15
15	$(\frac{12}{11})(\_\_\_\_) = 15$. $(15 \div \frac{12}{11} = \_\_\_\_\_\_)$	$(\frac{165}{12})$, $\frac{165}{12}$
16	$(\frac{2}{3})(\_\_\_\_) = 15$	$\frac{45}{2}$

The set of rational factors of 15 would be
17 a(an) ________ set. — infinite
(finite, infinite)

In fact, over the rational numbers every rational number except 0 would be a factor of 15. Thus, 73, for example, would also be a factor of 15,
18 since $73(\frac{\square}{73}) = 15$. — $\frac{15}{73}$

In the same way, every rational number except 0 would be a factor of every positive integer. Thus, factoring over the rational numbers will not be considered further. Usually, factoring over the positive integers gives us the most interesting results, and so when we speak of "factoring" a positive integer, we shall always mean factoring over the positive integers.

19 Which of the following lists contains only numbers which have no proper factors?

[A] 2, 3, 5, 7, 11, 13
[B] 3, 5, 7, 9, 11, 13
[C] 2, 3, 5, 7, 10, 11, 13, 15

There are no numbers in list [A] which have proper factors.
List [B] contains 9. List [C] contains 10 and 15.
9, 10, and 15 have proper factors.

The set {2,3,5,7,11,13,19} contains no numbers which
20 have ______ factors. — proper

Are there integers between 1 and 20 which have no proper factors and which are not in the
21 set {2,3,5,7,11,13,19}? ______ — yes

4, 6, 8, 10, 12, 14, and 16 do not belong since each of these numbers is divisible by 2 and thus
22 contain a proper factor of ______. — 2

9 and 15 do not belong since each contains a factor
23 of ______. — 3

24 ______ belongs in the set since it has no proper factors. — 17

25 { ________ } is the set of all positive integers greater than 1 and less than 20 which contain no proper factors.

{2,3,5,7,11, 13,17,19}

Numbers greater than 1 which have no proper factors are called prime numbers.

2 and 3 are prime numbers, [illegible]

26 are not ______ numbers.

27 17 ________ (is, is not) a prime number.

28 {2,3,7,11,13,17,19} contains [illegible] ______

A prime number is a positive integer [illegible] factor.

29 There are ________ (how many) prime [illegible]

30 The prime numbers less than [illegible] ______.

31 The next prime number after [illegible] ______.

A prime number is a positive [illegible]

32 than 1, and which has no ______ [illegible].

All even numbers greater than [illegible]

33 factor of ______, therefore, [illegible]

34 than 2 can be a ______ number.

The number 2 is a prime number. [illegible]

35 prime ______ number.

36 All positive odd numbers are prime.

[A] true

[B] false

Is 9 a prime number? [illegible] included as a prime number. [illegible]

When there is no possibility of [illegible] number simply as a "prime". Thus, [illegible] Rules of divisibility will be of some [illegible] is or is not prime.

37	9 + 3 = 12, therefore, 93 is divisible by ______.	3
38	93 ______ prime. (is, is not)	is not
	State whether each of the following is prime or not prime.	
39	105 ______ (prime, not prime)	not prime
40	3[illegible] ______	prime
41	13[illegible] ______	not prime
42	97 ______	prime
43	1[illegible] ______	prime
44	[illegible] ______	not prime

	7 is a prime factor of 14, since 7 is a prime	
45	and 7( ) = 14.	7(2) = 14
46	2 is also a prime ______ of 14.	factor
47	14 has ______ prime factors? (how many)	2
48	Every proper factor of 14 is also a ______ factor.	prime

49	On the other hand, 30 has ______ proper factors. (how many)	6
50	______ are the proper factors of 30.	2,3,5,6,10,15
51	Of these, the factors ______ are prime.	2,3,5
	If you were asked to write 30 as an indicated product of proper factors, you might write as possible answers:	
52	(2)( ),	(2)(15)
53	(3)( ),	(3)(10)
54	( )(6),	(5)(6)
	and (2)(3)(5).	
	Each of these may be called a factorization of 30. The last of these, (2)(3)(5), is of particular interest, since each factor in this product is a	
55	______ factor.	prime

We call this indicated product the prime factorization of 30.

56 Notice that, except for the order of the factors, there is only one prime ______ of 30. — factorization

This last result may be generalized:

Every non-prime positive integer except 1 has exactly one prime factorization.

This fact is often called the Fundamental Theorem of Arithmetic. The proof of this is beyond the scope of this course. We shall assume the truth of the theorem.

In view of the Fundamental Theorem of Arithmetic, it is true that for any positive integer which is different from 1 and is not prime, we can find the prime factorization.

To find the prime factorization of 63 we observe

57 that 63 is not even, thus ______ is not a factor. — 2

58 However, 63 is divisible by ______, the next smallest prime. — 3

59 Since ______ is the smallest prime factor of 63, — 3

60 let us divide 63 by 3. The quotient is ______. — 21

61 21 ______ prime. (is, is not) — is not

62 The smallest prime factor of 21 is ______. — 3

63 If we divide 21 by 3 the quotient is ______. — 7

64 7 ______ prime. (is, is not) — is

65 The prime factorization of 63 is ( )( )( ). (The order of the factors makes no difference.) — (3)(3)(7)

Find the prime factorization of each of the following:

66 15 = ___ • ___ — 3 • 5

67 8 = 2 • 2 • ___ — 2 • 2 • 2

68 54 = 2 • 3 • ___ • ___ — 2 • 3 • 3 • 3

69 100 = ___ • ___ • ___ • ___ — 2 • 2 • 5 • 5

70 24 = ______ — 2 • 2 • 2 • 3

2

Let us find the prime factorization of 3276 making full use of our tests for divisibility.

71	We know that 2 divides 3276, since the last digit, 6, is divisible by ______.	2
72	Divide 3276 by 2. The quotient is ______	1638
73	2 is also a factor of 1638. Divide 1638 by 2. The quotient is ______.	819
	Since 2 is not a factor of 819, we apply our test for divisibility by 3:	
74	$8 + 1 + 9 =$ ______, which is divisible by 3.	18
	Hence, 819 is divisible by the prime number 3.	
75	Dividing 819 by 3 gives the quotient ______.	273
76	3 ______ divide 273. (does, does not)	does
77	Divide 273 by 3. The quotient is ______.	91
78	91 is not divisible by 3, since $9 + 1 =$ ______, which is not divisible by 3.	10
79	91 is not divisible by 5, since it does not end in 0 or ______.	5
80	Trying the next largest prime after 5, we see that ______ does divide 91.	7
81	In fact, 91 divided by 7 gives the quotient ______, which is a prime number.	13
82	Summarizing, $3276 = (2)(\quad)$	2(1638)
	$= (2)(2)(819)$	
	$= (2)(2)((3)(273))$	
83	$= (2)(2)(3)((\quad)(91))$	(2)(2)(3)((3)(91))
84	$= (2)(2)(3)(3)((\quad)(13))$	(2)(2)(3)((7)(13))
	$= (2)(2)(3)(3)(7)(13)$	
	This last product is the prime factorization of 3276.	

Writing this whole process is rather clumsy; a more compact way to write it is the following.

[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	[illegible]
[illegible]	7
43	43
1	

Here the smallest prime factor at any stage is to the right of the line, and the successive quotients are shown beneath each other. The process is continued until a quotient of 1 is obtained. Then the prime factorization can be read off from the factors in the column on the right.

85 Suppose that the process has been carried out for 258:

258	2
129	3
43	43
1	

The prime factorization of 258 is

[A] 2 • 3 • 43 [C] 2 • 3 • 43 • 1

[B] 3 • 43 • 2 [D] all of these

It is true that 2 • 3 • 43, 3 • 2 • 43, and 2 • 3 • 43 • 1 all name 258. We must remember, however, that 1 is not a prime. Hence, [C] is not correct. The order of the factors makes no difference. Hence, either [A] or [B] is correct.

Find the smallest prime factor of each of the following numbers. Insofar as possible, use the rules of divisibility which you have learned.

86 115 ______ 5

87 135 ______ 3

88 321 ______ 3

89 484 ______ 2

90 539 ______ 7

91 143 ______ 11

Find the prime factorization of the following numbers:

92	98 = ______	2 · 7 · 7
93	432 = ______	2·2·2·2·3·3·3
94	625 = ______	5 · 5 · 5 · 5
95	149 = ______	149 (149 is prime)
96	825 = ______	3 · 5 · 5 · 11

We have seen in a number of examples a process for finding the prime factorization of an integer. The process works for any integer. For some integers, however, it is not the simplest way of proceeding.

To find the prime factorization of 90, we might, as in the preceding examples, write:

$$\begin{array}{r|l} 90 & 2 \\ 45 & 3 \\ 15 & 3 \\ 5 & 5 \\ 1 & \end{array}$$

97 We conclude: 90 = ( )( )( )( ). — (2)(3)(3)(5)

It is simpler in this case, however, to think:

98 $90 = (\quad)(10)$ — (9)(10)

99 $\quad = ((3)(3))((\quad)(\quad))$ — ((3)(3))((2)(5))

We conclude: 90 = (3)(3)(2)(5).

Comparing this factorization with the one in Item 97, we observe that, although the order of the factors is different, we have obtained essentially the same result by each process.

That is, we have found that in the prime factorization of 90:

2 is used as a factor once;

100 3 is used as a factor ______; — twice

101 5 is used as a factor ______. — once

The Fundamental Theorem of Arithmetic is illustrated by this example. Two different ways of finding the prime factorization always lead to the same factorization, although possibly the order of the factors is different.

Find the prime factorization of 1764.

Since the number represented by the last two digits of

102 1764 is divisible by ______, — 4

103 $1764 = \underline{(\ \ )(441)}$ — 4(441)

104 $= (2)(2)\underline{((\ \ )(49))}$. Since $4 + 4 + 1 = 9$, 9 divides 441. — (9)(49)

105 = ______. — (2)(2)(3)(3)(7)(7)

Find the prime factorization of each, using any convenient method.

106 $486 = \underline{(\ \ )(81)}$ 2 and 3 divide 486. — (6)(81)

107 = ______. — (2)(3)(3)(3)(3)(3)

108 4840 = ______ — (2)(5)(11)(11)(2)(2)

109 1455 = ______ — (3)(5)(97)

110 1096 = ______ — (2)(137)(2)(2)

111 Suppose we know that x, y, and z are positive integers such that

$x \cdot y \cdot z = 66$ and that

$x \neq 11$, $y = 2$; then we can conclude that

z has which of the following values?

[A] 3 [B] 11 [C] either 3 or 11

The prime factorization of 66 is (2)(3)(11). One of the numbers x, y, or z must be 11 because the prime factorization of 66 is unique. Since $x \neq 11$ and $y = 2$, it follows that $z = 11$. [B] is the correct choice.

In finding the prime factorization of 300 we might think:

112 $300 = (2)(\_\_\_\_)$, — 150

or we might think:

113 $300 = (3)(\_\_\_\_)$ — 100

The prime factorization we find will be the same in either case (except for the order of the factors, which doesn't matter).

From Item 112 we can conclude that the prime factorization includes the prime number 2 as a factor at least once.

Likewise, from Item 113 we conclude that the prime
114 number ______ appears in the prime factorization of 300. — 3

Thus, if we know only that 2 and 3 are both factors of 300, we can conclude that the prime factorization of 300 has this general form:

$$300 = (2)(3)(\underbrace{\qquad\qquad}_{\text{some primes}})$$

Hence, if we know only that 2 and 3 are factors of 300, we can conclude: (2)(3), or 6, is also a factor of 300.

In fact, precisely the same reasoning shows that if any integer n is divisible by 2 and 3, then it is
115 divisible by ______. — 6

Likewise, we can conclude that if 2 and 5 are both
116 factors of a number, then so is (2)( ), or 10. — (2)(5)

Moreover, if 4 is a factor of a number, then we can conclude that, in the prime factorization of the number,
117 2 appears as a factor at least $\underset{\text{(how many times)}}{\underline{\qquad\qquad}}$. — twice

Thus, if 4 and 3 are both factors of a number n, then the prime factorization of n has the form

$$n = (2)(2)(3)(\underbrace{\qquad\qquad}_{\text{some primes}})$$

Hence, such a number n must have 12 as a factor.

---

118 If a number n is divisible by 6 and 3, can we conclude that it is divisible by 18?

[A] yes [B] no

If a number n is divisible by 6, its prime factorization contains (2)(3). Knowing that the number n is divisible by 3 gives no additional information. Hence, [B] is correct. Notice that 24 is divisible by 6 and by 3, but not by 18.

119	Is 15 a factor of 91,215 ?	yes[test for 3; 5]
120	Is 12 a factor of 187,326,648 ?	yes[test for 3; 4]
121	Is 18 a factor of 187,326,648 ?	yes[test for 2; 9]

* There is an interesting observation which may be made about factorization of positive integers.

We shall restrict ourselves to writing an integer as the product of exactly two integers. (They are not necessarily primes.)

We may write 6 as a product of two factors in two
*122 distinct ways: 1×6, and _______. — 2×3

*123 The sum of 1 and 6 is _______. — 7

*124 The sum of 2 and 3 is _______. — 5

There are three distinct ways of writing 12 as a
*125 product of two factors: $1 \cdot 12$, $2 \cdot 6$ and _______. — $3 \cdot 4$

*126 The sum of 1 and 12 is _______. — 13

*127 The sum of 2 and 6 is _______. — 8

*128 The sum of 3 and 4 is _______. — 7

100 has five distinct factorizations into a product of two factors.

*129 The sum of 1 and 100 is _______. — 101

*130 The sum of 2 and 50 is _______. — 52

*131 The sum of 4 and 25 is _______. — 29

*132 The sum of 5 and 20 is _______. — 25

*133 The sum of 10 and 10 is _______. — 20

Notice that, in our examples, the largest sum for each occurs for the factorization $1 \times n$. Do you think that this is true for every positive integer?

We shall learn more about the sums of factors in Section 12-4.

12-4. Some Facts About Factors

In this section we are going to discover some interesting relationships between factors of numbers and factors of sums of these numbers. We shall make some conjectures and then try to prove them.

First let us look at a special case involving the factor 2.

1	2 ______ (is, is not) a factor of 10, 2 ______ (is, is not) a factor of 16.	is, is
2	Is 2 a factor of (10 + 16)? ______	yes
3	2 ______ (is, is not) a factor of 8, 2 ______ (is, is not) a factor of 32.	is, is
4	Is 2 a factor of (8 + 32)? ______	yes
	A conjecture which we may make is:	
5	If 2 is a ______ of two numbers, then 2 is a	factor
6	factor of their ______.	sum
	Is this conjecture true? Let's see.	

	Recall that "a is a factor of b" means "there is some integer q such that $b = aq$".	
	Thus, if 2 is a factor of c, then	
7	$c = \_\_ q$.	$c = 2q$
	If r is an integer, and if $c = 2r$, then	
8	2 is a ______ of c.	factor
9	2r is an ______ number since it has a factor of 2.	even

We are going to prove: For positive integers b and c, if 2 is a factor of both b and c, then 2 is a factor of b + c.

Since this statement expresses a relationship involving factor (a multiplicative idea) and the sum of two numbers, we should expect to use the distributive property in the proof.

	Proof:		
10	There exists an integer p such that b = 2p.	______ is a factor of b.	2
11	There exists an integer q such that c = ______.	2 is a factor of c.	c = 2q
	b + c = 2p + 2q	Addition property of equality.	
12	Hence, b + c = 2(______).	Distributive property.	2(p + q)
	(p + q) is an integer.	The set of integers is closed under addition.	
	Hence, 2 is a factor of (b + c).	Definition of factor.	

13 We have proved that

[A] the sum of two odd numbers is even.

[B] the sum of an odd number and an even number is odd.

[C] the sum of two even numbers is even.

Although [A] and [B] are true, this is not what we proved.
[C] is correct.

We conjectured and proved that if 2 is a factor of two numbers, then 2 is a factor of their sum. We now ask ourselves, would it be true for other factors?

14	3 ______ (is, is not) a factor of 6, 3 ______ (is, is not) a factor of 9.	is, is
15	Is 3 a factor of (6 + 9)? ______	yes
16	7 ______ (is, is not) a factor of 14, 7 ______ (is, is not) a factor of 35.	is, is
17	Is 7 a factor of (14 + 35)? ______	yes
18	13 is a factor of both 39 and 52. ( )(13) = 39	(3)(13)
19	and ( )(13) = 52.	(4)(13)
20	13 is a factor of (39 + 52) since 13( ) = 91.	13(7)

At this point you may suspect what the general statement will be.

<u>Theorem 12-4a.</u> For positive integers a, b, and c, if a is a factor of both b and c, then a is a factor of b + c.

Try to prove Theorem 12-4a on a separate sheet of paper. Hint: The proof is similar to the previous one. Do your best. Use Items 21-24 with which to compare your proof.

<u>Theorem 12-4a.</u> For positive integers a, b, and c, if a is a factor of both b and c, then a is a factor of b + c.

<u>Proof</u>:

Item	Statement	Reason	Answer
21	There exist integers p and q such that b = ap and c = aq.	It is given that a is a factor of b and also of ______.	c
22	b + c = ap + aq	Addition property of ______.	equality
23	b + c = a(p + q)	______ property.	Distributive
24	(p + q) is an integer.	The set of integers is ______ under addition.	closed
	Therefore, a is a factor of b + c.	Definition of factor.	

Two other useful theorems will be considered. The following items suggest one of them.

Item	Question	Answer
25	2 ______ (is, is not) a factor of 6, and 2 ______ (is, is not) a factor of (6 + 5).	is, is not
26	Is 2 a factor of 5? ______	no
27	5 ______ (is, is not) a factor of 10, and 5 ______ (is, is not) a factor of (10 + 13).	is, is not
28	Is 5 a factor of 13? ______	no
29	8 ______ (is, is not) a factor of 24, and 8 ______ (is, is not) a factor of (24 + 31).	is, is not
30	8 ______ (is, is not) a factor of 31.	is not

31 Our suspicion is: If a number is a factor of the first of two numbers and if it is not a factor of their sum, then it ________ a factor of the second number. (is, is not) | is not

The theorem suggested by Items 29-31 is:

> <u>Theorem 12-4b.</u> For positive integers a, b, and c, if a is a factor of b, and a is <u>not</u> a factor of (b + c), then a is <u>not</u> a factor of c.

To prove this theorem we shall use an indirect proof. Recall from the end of Section [illegible] our discussion of indirect proof. In an indirect proof if we "assume" that the conclusion is false, and if we are led to a <u>contradiction</u>, we are sure the conclusion must be true.

Restating our theorem: For positive integers a, b, and c, if a is a factor of b, and a is <u>not</u> a factor of (b + c), then a is <u>not</u> a factor of c.

<u>Proof</u>:

There are two possibilities: Either a is a factor

32 of c or a ________ a factor of c. | is not

33 We wish to prove that a ________ a factor of c. (is, is not) | is not

Assume a <u>is</u> a factor of c.

Since the theorem states that a is a factor of b and we have assumed a is a factor of c, we can say

34 that a is a factor of both ______ and ______. | b, c

35 Hence, a is a factor of (____). (See Theorem 12-4a). | b + c

But we were given that a is <u>not</u> a factor of (b + c).

We have been led to a contradiction.

Our assumption that a <u>is</u> a factor of c must be

36 ________. (false, true) | false

Therefore, a is <u>not</u> a factor of c.

Complete the next five items and see if they lead you to another conclusion.

37 3 ________ (is,is not) a factor of 9, and 3 ________ (is,is not) a factor of (9 + 12). is, is

38 Is 3 a factor of 12? ______ yes

39 7 ________ (is,is not) a factor of 21, and 7 ________ (is,is not) a factor of (21 + 28). is, is

40 7 ________ (is,is not) a factor of 28. is

Our suspicion is: If a number is a factor of the first of two numbers and if it is a factor of their sum, then

41 it ________ (is,is not) a factor of the second. is

Items 37-41 suggest

Theorem 12-4c. For positive integers a, b, and c, if a is a factor of b, and a is a factor of (b + c), then a is a factor of c.

*42. If you wish, try to prove Theorem 12-4c and then turn to page i to see one method of proving it.

Although these theorems will be used in a later chapter, an application of them will be seen in the following problem.

The area of a rectangular field is 288 square feet. One-half of its perimeter is 34 feet. Find the length and width of the field. (Recall that the area of a rectangle is equal to the product of the length and width, and one-half the perimeter is equal to the sum of the length and width.)

Undoubtedly, you can find the solution by trying pairs of numbers until you find a pair whose sum is 34 and whose product is 288. We shall, however; use some of the theorems we have proved to guide our reasoning.

	We wish to find two integers whose product is 288 and whose sum is 34.	
43	The prime factorization of 288 is _______.	(2)(2)(2)(2)(2)(3)(3)
44	288 contains 2 as a factor ______ times and 3 as	five
45	a factor _______ times.	two
46	Between them the two integers must contain in total five factors of 2 and ______ factors of 3.	two
47	If one integer contained the factor (2)(2), then the other integer must contain the factors _______.	(2)(2)(2)(3)(3)
48	Although the product of (2)(2) and (2)(2)(2)(3)(3) is 288, (2)(2) and (2)(2)(2)(3)(3) is not the solution to our problem since the sum of 4 and 72 is not _______ as required.	34

Since the product of the two integers contains factors of 2, we know that at least one of the integers must contain a factor of 2 which we represent by this diagram:

$$\underbrace{(2)(\ )}_{\text{one integer}} + \underbrace{(\ )}_{\text{other integer}} = 34$$

49	34 is an ________ (even, odd) number and thus contains	even
50	_______ as a factor.	2
51	By Theorem 12-4c, since 2 is a factor of at least one of the integers and 2 is a factor of 34, the sum of the two integers, 2 __________ (is, is not) a factor of the other integer.	is

We represent our reasoning to this point as

$$\underbrace{2(\ )}_{\text{one integer}} + \underbrace{2(\ )}_{\text{other integer}} = 34.$$

52	288 has 5 factors of 2, consequently, we have ________ (how many) more factors of 2 for which to account.	three
	Let's answer next the question whether each of the integers may contain another factor of 2.	

If each of the integers contained another factor of 2, each would now contain at least two factors of 2 and then would be divisible by ______. — 4

But if 4 or (2)(2) is a factor of each of the two integers, then 4 ______ (would, would not) be a factor of their sum 34, by Theorem 12-4a. — would

However, 4 is not a factor of 34. Therefore, one of the integers cannot contain more than one factor of ______, and the other integer contains the remaining four factors of 2. — 2

The representation of the distribution of the factors of 2 is as follows:

$$\underbrace{\text{one integer}}_{(2)(\ )} \quad + \quad \underbrace{\text{other integer}}_{(2)(2)(2)(2)(\ )}$$

We must now decide whether the two factors of 3 must be split between the two integers or whether both factors of 3 must be contained in one of them.

If 3 were a factor of both of the integers, by Theorem 12-4a, 3 ______ (would, would not) be a factor of 34. — would

Since 3 ______ (is, is not) a factor of 34, 3 is not a factor of both integers, by Theorem 12-4b. — is not

Hence, the factors of 3 must be kept together.

We have the following two choices:

$$\underbrace{\text{one integer}}_{2} \quad + \quad \underbrace{\text{other integer}}_{(2)(2)(2)(2)(3)(3)} = 34$$

$$\underbrace{\text{one integer}}_{(2)(3)(3)} \quad + \quad \underbrace{\text{other integer}}_{(2)(2)(2)(2)} = 34$$

But $(2)(2)(2)(2)(3)(3) = 2 + 144$ and $2 + 144 \neq 34$.

The correct choice is

$(2)(3)(3) + (2)(2)(2)(2) =$ ______ $+ 16$ — 18

$= 34.$

41

	At last! We see that the rectangle which has an area of 288 sq. ft. and which has 34 feet as half of	
59	its perimeter, must have a length of ______ feet and	18
60	a width of ______ feet.	16
61	$18 \times 16 =$ ______	288
62	$18 + 16 =$ ______	34

Although the reasoning which we have outlined is lengthy, it is not difficult.

In the following exercises try to apply the theorems presented in this section or to recognize when they are applied.

63	The prime factorization of 36 is ______.	$2 \times 2 \times 3 \times 3$
	In order to find two numbers whose product is 36 and whose sum is divisible by 3 but not by 2, we should	
64	split the ______ between the two numbers, but keep	3's
65	the ______ together.	2's
66	Therefore, the numbers are ______ and ______.	$(2\times2\times3)$ and 3, or 12 and 3
67	Two numbers whose product is 36 and whose sum is divisible by 2 but not by 3, are ______ and ______.	2 and $2\times3\times3$, or 2 and 18
68	There are two pairs of numbers whose product is 36 and whose sum is divisible by neither 2 nor 3. They are 1 and 36, and ______ and ______.	3×3 and 2×3, or 9 and 4

69	The prime factorization of 150 is ______.	$2 \times 3 \times 5 \times 5$
	Can you find two integers whose product is 150 and	
70	whose sum is even? ______ (yes,no)	No [2 occurs only once in the factorization.]
71	whose sum is divisible by 5? ______ (yes,no)	Yes [we can split the 5's]
72	whose sum is divisible by 3? ______ (yes,no)	No [3 occurs only once in the factorization.]

73 How many pairs of integers have 150 as their product and have a sum which is not divisible by 5?

[A] one pair

[B] two pairs

[C] more than two pairs

We must keep the 5's together, because the sum is not to be divisible by 5. We could have:

$2 \cdot 3 \cdot 5 \cdot 5$ and 1
$3 \cdot 5 \cdot 5$ and 2
$2 \cdot 5 \cdot 5$ and 3
$5 \cdot 5$ and 6

Thus, there are 4 pairs of numbers whose product is 150 and whose sum is not divisible by 5. The correct answer is [C].

*74 Write the prime factorization of the first number in each of the following. Use it to find two numbers whose product and whose sum are as indicated. One of these is impossible. Which one is it?

[A] Product is 216 and sum is 217.

[B] Product is 330 and sum is 37.

[C] Product is 500 and sum is 62.

The correct choice is [C]. $500 = 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5$; 62 is an even number not divisible by 5; therefore, the grouping would have to be 2 and $2 \cdot 5 \cdot 5 \cdot 5$. But $2 + 250 \neq 62$.

Factorizations for [A] and [B] are:

$$216 = 1 \times 216, \quad 1 + 216 = 217$$
$$330 = 15 \times 22, \quad 15 + 22 = 37$$

43

*75 If 4 boys shovel snow from sidewalks and charge 50 cents for a store and $1.50 for a house, how many store walks and how many house walks should they shovel in order to split the money evenly?

[A] An even number of store jobs and an even number of house jobs.

[B] An odd number of store jobs and an odd number of house jobs.

[C] Either [A] or [B].

If x is the number of store jobs and y is the number of house jobs, then $50x + 150y$ is the number of cents earned.

$$50x + 150y = 50(x + 3y)$$

Since 2 divides 50 and since 4 must divide the expression $50(x + 3y)$, we should select even values of $x + 3y$. If x is even, then $3y$ must be even. $3y$ is even when y is even.

If x is odd, then $3y$ must be odd, since if a sum of two integers is even, the integers must be either both even or both odd. $3y$ is odd when y is odd.

Thus, if the boys accept an even number of store jobs, they must accept an even number of house jobs; if they accept an odd number of store jobs, they must accept an odd number of house jobs. [C] is the correct choice.

* For what positive integer x is 3 a factor of $6 + 4x$?

Theorem 12-4a states: For positive integers a, b, and c, if a is a factor of both b and c, then a is a factor of $b + c$.

*76 $6 + 4x$ corresponds to ______ in the theorem. — $b + c$

*77 3 is a factor of ______. — 6

Hence, 3 will be a factor of $6 + 4x$ if it is also

*78 a factor of ______. — $4x$

Therefore, 3 is a factor of $6 + 4x$ if x is any

*79 multiple of ______. — multiple of 3

We find factoring to be useful in finding simpler names for some fractions.

80	$\frac{18}{24}$ may be written as $\frac{(9)(2)}{(12)(\ \ )}$ where	$\frac{(9)(2)}{(12)(2)}$
81	______ is a factor common to both 18 and 24.	2
82	$\frac{18}{24} = \frac{9(2)}{12(2)} = \frac{9}{12} \cdot \frac{2}{2} = \frac{9}{12} \cdot$ ______ $= \frac{9}{12}$.	$\frac{9}{12} \cdot 1$
83	$\frac{18}{24}$ may also be written as $\frac{(6)(\ \ )}{(8)(3)} = \frac{6}{8}$.	$\frac{(6)(3)}{(8)(3)}$
84	3 is a common ______ of both 18 and 24.	factor
85	You have probably recognized that although 2 and 3 are ______ factors of 18 and 24, neither is the greatest common factor.	common
86	The ______ common factor of 18	greatest
87	and 24 is ______.	6
88	Our experience from arithmetic enabled us to easily see that 6 is the ______ ______ ______ of 18 and 24.	greatest common factor

Had we not recognized the greatest common factor from earlier experience we could have found it as follows.

89	Write the set of all factors of 18. ______	{1,2,3,6,9,18}
90	{ ______ } is the set of all factors of 24.	{1,2,3,4,6,8,12,24}
	The set of factors common to both 18 and 24 is the intersection of {1,2,3,6,9,18} and {1,2,3,4,6,8,12,24}.	
91	$\{1,2,3,6,9,18\} \cap \{1,2,3,4,6,8,12,24\} = \{$ ______ $\}$. (Recall that $\cap$ is the intersection symbol.)	{1,2,3,6}
92	Look at {1,2,3,6}. It is the set of all ______ factors of 18 and 24.	common
93	______ is the greatest common factor.	6

94	{____} is the set of all factors of 45, and	{1,3,5,9,15,45}
95	{____} is the set of all factors of 60.	{1,2,3,4,5,6,10, 12,15,20,30,60}
	The intersection of the set of factors of 45 and	
96	the set of factors of 60 is {____}.	{1,3,5,15}
97	{1,3,5,15} is the set of all ______ common to 45 and 60.	factors
98	______ is the greatest common factor of 45 and 60.	15

	Find the greatest common factor of the following:	
99	32 and 56 ______	8
100	9 and 15 ______	3
101	21 and 70 ______	7
102	16, 24, and 36 ______	4

12-5. Summary

We considered factorization in the set of positive integers.

The positive integer m is a factor of the positive integer n if $mq = n$, where q is a positive integer. If m does not equal 1 or n, we say that m is a proper factor of n.

A prime number is a positive integer greater than 1 which has no proper factors.

If n is a positive integer greater than 1, then either:

n is a prime number; or
n can be written as a product of primes (prime factorization).

The Fundamental Theorem of Arithmetic states that there is only one prime factorization for a given positive integer. The order in which we write the prime factors makes no difference.

Tests for divisibility of a number:

A number is divisible

by 2 if the last digit of the number is even.

by 3 if the sum of the digits is divisible by 3.

by 4 if the number represented by the last two digits is divisible by 4.

by 5 if the last digit is 0 or 5.

by 6 if the number is divisible by both 2 and 3.

by 8 if the number represented by the last three digits is divisible by 8.

by 9 if the sum of the digits is divisible by 9.

There is no easy test for divisibility by 7.

The following theorems were proved:

For positive integers a, b, and c, if a is a factor of both b and c, then a is a factor of (b + c).

For positive integers a, b, and c, if a is a factor of b, and a is <u>not</u> a factor of (b + c), then a is <u>not</u> a factor of c.

For positive integers a, b, and c, if a is a factor of b, and a is a factor of (b + c), then a is a factor of c.

The greatest common factor of two numbers is the greatest factor common to both numbers.

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Chapter 13

FRACTIONS

13-1. Multiplication of Fractions

From previous work in arithmetic, we are already familiar with the process of multiplying fractions when these are numbers of arithmetic. For example, we know that

$$\frac{3}{8} \cdot \frac{7}{2} = \frac{3 \cdot 7}{8 \cdot 2}.$$

In Chapter 8, multiplication was defined for the real numbers. By the definition, we have, for example,

1 $\frac{3}{8}(-\frac{7}{2}) = -(\frac{3}{8} \cdot \frac{7}{2}) =$ _______, $-\frac{21}{16}$

2 and $(-\frac{3}{8})(-\frac{7}{2}) =$ ____ $\cdot \frac{7}{2} = \frac{21}{16}$ $\frac{3}{8} \cdot \frac{7}{2}$

So each of the products involving real numbers is expressible in terms of non-negative numbers and possible taking of opposites. Furthermore, examples such as $\frac{3}{8} \cdot \frac{7}{2}$ suggest a theorem which we can now prove for all real numbers.

Theorem 13-1. For any real numbers a, b, c, d, if $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

To prove the theorem, notice first that $\frac{1}{b}$ and $\frac{1}{d}$

3 are real numbers, since $b \neq 0$ and _______. $d \neq 0$

4 $\frac{a}{b} \cdot \frac{c}{d} = (a \cdot \frac{1}{b})(c \cdot \frac{1}{d})$ by the definition of _______. division

5 $= (ac)(\frac{1}{b} \cdot \quad)$ using both the commutative and associative properties of multiplication. $(ac)(\frac{1}{b} \cdot \frac{1}{d})$

6 $= (ac)(\frac{1}{bd})$ since the product of the reciprocals equals the _______ of the product. reciprocal

7 $= \frac{ac}{bd}$ by the _______ of division. definition

	Use the theorem $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ to multiply the following fractions:	
8	$\frac{4}{7} \cdot \frac{3}{5} =$ ______	$\frac{12}{35}$
9	$\frac{3}{8} \cdot \frac{-7}{2} =$ ______	$\frac{-21}{16}$
10	$\frac{3}{8} \cdot \frac{7}{-2} =$ ______	$\frac{21}{-16}$
11	$\frac{4}{3y} \cdot \frac{2x}{5} =$ ______, $y \neq 0$	$\frac{8x}{15y}$
12	$\frac{2x}{5} \cdot \frac{4x}{11} =$ ______	$\frac{8x^2}{55}$

Notice that the second factor in Item 9 is $\frac{-7}{2}$ and that the second factor in Item 10 is $\frac{7}{-2}$. We can show that both of these name the same number. In fact, if a and b are real numbers $(b \neq 0)$, we can show that $\frac{-a}{b}$, $\frac{a}{-b}$, $-\frac{a}{b}$ all name the same number.

	To show that $\frac{-a}{b} = -\frac{a}{b}$, note that	
13	$\frac{-a}{b} = -a(\frac{1}{b})$ by the definition of ______.	division
14	$= -(a \cdot$ ___ since $(-x)y = -xy$.	$-(a \cdot \frac{1}{b})$
15	$= -\frac{\square}{\square}$ by the ______ of division.	$-\frac{a}{b}$, definition

Similarly we can show that $\frac{a}{-b} = -\frac{a}{b}$. Try to construct the proof for yourself and compare your results with those on page 1.

Thus we see that Items 9 and 10 ask for the same product as Item 1. Of the three forms $\frac{-a}{b}$, $\frac{a}{-b}$, $-\frac{a}{b}$, we shall agree that $-\frac{a}{b}$ is the simplest name for the number.

	A common name is, in a sense, the simplest name for a number. For example, $\frac{2}{-3}$, $\frac{-2}{3}$, $-\frac{2}{3}$ all name the same number; the common name of this number is	
16	______.	$-\frac{2}{3}$

Recall that on previous occasions we referred to some special names for numbers which we called "common names". We noticed before, for example, that $\frac{2}{3}$, $\frac{4}{6} = \frac{2 \cdot 2}{3 \cdot 2}$, $\frac{6}{9} = \frac{2 \cdot 3}{3 \cdot 3}$, ... , all name the same number, and that if $k \neq 0$, $\frac{2k}{3k} = \frac{2}{3}$. This illustrates the general statement that if $k \neq 0$, $\frac{ak}{bk} = \frac{a}{b}$ whenever a and b are real numbers and $b \neq 0$. This statement is in agreement with Theorem 13-1, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$, if we let $c = k$ and $d = k$.

	If $c = k$ and $d = k$, we have	
17	$\frac{\square k}{\square k} = \frac{a}{b} \cdot \frac{k}{k}$	$\frac{ak}{bk} = \frac{a}{b} \cdot \frac{k}{k}$
18	$= \frac{a}{b} \cdot 1$, since $\frac{k}{k} =$ ______ for any $k \neq 0$.	$\frac{k}{k} = 1$
19	$= \frac{a}{b}$ by the ______ property of 1.	multiplication

Making use of the fact that $\frac{ak}{bk} = \frac{a}{b}$, we can simplify fractions. For example, since $\frac{6}{21} = \frac{2 \cdot 3}{7 \cdot 3}$, the fractions $\frac{6}{21}$ can be simplified to $\frac{2}{7}$. However, when we are asked to simplify a given expression, it is important that we understand exactly what is meant. "Simplify" means "find the common name for". We recall three important ideas, or conventions, regarding common names.

1. A common name contains no indicated division if it can be avoided. For example, $\frac{15}{3}$ should be "simplified" to 5.
2. If a common name must contain an indicated division, then the resulting expression should be written in "lowest terms". For example, $\frac{6}{9}$ should be changed to $\frac{2}{3}$ if we want the common name.
3. We prefer writing $-\frac{a}{b}$ to either of the forms $\frac{-a}{b}$ or $\frac{a}{-b}$.

We have defined a "fraction" as a symbol which indicates the quotient of two numbers. Thus, a fraction involves two numerals, a numerator and a denominator. When there is no possibility of confusion, we shall use the word "fraction" to refer also to the number which is represented by the fraction. When there is a possibility of confusion, we must go back to our strict meaning of fraction as a numeral.

We shall continue our agreement that the domains of the variables in a fraction exclude those values

20 which make the _______ equal to 0. denominator

If $y = 0$, then $\frac{x}{y}$, which means $x(\frac{1}{y})$, is not

21 a number since _______ has no reciprocal. 0

In $\frac{y}{y - 2}$, the domain of y is the set of all real

22 numbers except _______. 2 [Notice that 0 is in the domain.]

23 In $\frac{xy - y}{y(x - 1)}$, $x \neq$ _______ and $y \neq$ _______. $x \neq 1$, $y \neq 0$

Simplify $\frac{3y - 3}{2(y - 1)}$, $y \neq 1$,

24 $\frac{3y - 3}{2(y - 1)} = \frac{3(y - 1)}{2(y - 1)}$ by the _______ property. distributive

25 $= \frac{3}{2}$, $\frac{ak}{bk} =$ _______, if $k \neq 0$, $b \neq 0$. $\frac{ak}{bk} = \frac{a}{b}$

We can find a "simpler name" for $\frac{4}{6}$, but not for $\frac{2}{3}$. Similarly $\frac{2a}{b}$ cannot be simplified.

Find the common name for each of the following:

26 $\frac{7x + 7}{7} =$ _______ $x + 1$

27 $\frac{7x + 1}{7} =$ _______ $\frac{7x + 1}{7}$

28 $\frac{2x - 3}{2x - 3} =$ _______, $x \neq$ _______ 1, $x \neq \frac{3}{2}$

29 $\frac{2x - 3}{2y - 3} =$ _______, $y \neq$ _______ $\frac{2x - 3}{2y - 3}$, $y \neq \frac{3}{2}$

We notice that when there is a common factor k in the numerator and the denominator, then $\frac{a}{b}$ is a simpler name for $\frac{ak}{bk}$. Further, if a and b have no common factor other than 1, then the fraction $\frac{a}{b}$ is the simplest name for the number.

Simplify:

$\frac{3a^2b}{5aby}$, $a \neq 0$, $b \neq 0$, $c \neq 0$

30 $\frac{3a^2b}{5aby} = \frac{3a(ab)}{5y(ab)}$ by associative and ______ properties of multiplication. — commutative

31 $= \_\_\_\_\_$, $\frac{ak}{bk} = \frac{a}{b}$ if $k \neq 0$, $b \neq 0$. — $\frac{3a}{5b}$

Simplify $\frac{(2x + 5) - (5 - 2x)}{-8}$

32 $\frac{(2x + 5) - (5 - 2x)}{-8} = \frac{2x + 5 - 5 + \square}{-8}$ — $2x$

33 $= \frac{\square}{-8}$ — $\frac{4x}{-8}$

$= \frac{4x}{4(-2)}$

34 $= \_\_\_\_\_ = -\frac{x}{2}$ — $\frac{x}{-2}$

Simplify:

35 $\frac{8(1 - b) + 2}{5 - 4b} = \_\_\_\_, b \neq \_\_\_\_$ — 2, $b \neq \frac{5}{4}$

36 $\frac{3}{5} \cdot \frac{7}{6} = \frac{3 \cdot 7}{5 \cdot 6} = \_\_\_\_\_$ — $\frac{7}{10}$

37 $\frac{x}{3} \cdot \frac{5}{6} = \frac{\_\_ \cdot 5}{3 \cdot \_\_} = \_\_\_\_\_$ — $\frac{x \cdot 5}{3 \cdot 6} = \frac{5x}{18}$

38 $(-\frac{9}{8})(-\frac{7}{2}) = \_\_\_\_\_$ — $\frac{63}{16}$

39 $(-2)\frac{5}{9} = (\frac{-2}{1})\frac{5}{9} = \frac{-10}{9} = \_\_\_\_\_$ — $-\frac{10}{9}$

40 $\frac{4}{7} \cdot \frac{21}{10} = \_\_\_\_\_$ — $\frac{6}{5}$

For Item 40, notice that the fraction $\frac{4 \cdot 21}{7 \cdot 10}$ can be rewritten as $\frac{(2 \cdot 2)(3 \cdot 7)}{7(2 \cdot 5)}$ which in turn can be written as $\frac{2 \cdot 3(2 \cdot 7)}{5(2 \cdot 7)}$.

Simplify:

41 $(\frac{3}{2})(\frac{14}{9}) = \frac{3\cdot2\cdot\_\_\_\_}{\_\_\_\_\_\_} = \frac{7}{3}$ — $\frac{3\cdot2\cdot7}{2\cdot3\cdot3}$

42 $\frac{9}{51} \cdot \frac{78}{3} = \_\_\_\_\_\_ = \_\_\_\_\_\_$ — $\frac{3\cdot3\cdot2\cdot3\cdot13}{3\cdot17\cdot3} = \frac{78}{17}$

By displaying the numbers in the numerator and denominator in factored forms, it is easier to see whether the fraction we will get is in "lowest terms".

Simplify each of the following where possible. Indicate the restrictions on the domains of the variables. The answers are on page ii.

43. $\frac{2(x - 2)}{3(x - 2)}$

44. $\frac{2x - 4}{3x - 6}$

45. $\frac{xy + y}{x + 1}$

46. $\frac{3x + 6}{3}$

47. $\frac{3}{4} \cdot \frac{x + 2}{3}$

48. $\frac{n + 3}{2} \cdot \frac{n + 2}{3}$

49. $\frac{n + 3}{2} \cdot \frac{2}{n + 3}$

50. $(4a^2)(\frac{3}{a})$

51. $\frac{(4t - 5) - (t + 1)}{3}$

52. $\frac{3x - 9}{2} \cdot \frac{4}{x - 3}$

53. $\frac{xy + y}{x - 1} \cdot \frac{xy - y}{x + 1}$

54. $\frac{3(a - 5)}{5(5 - a)}$

Did you get the correct response for Item 54? Recall that $-(a - 5) = -a + 5 = 5 - a$. Let's try a few more examples involving this situation.

Simplify:

55 $\frac{2x - 4}{6 - 3x} = \frac{2(x - 2)}{\square(x - 2)} = \_\_\_\_\_\_, \ x \neq 2$ — $\frac{2(x-2)}{-3(x-2)} = -\frac{2}{3}$

56 $\frac{2a - a^2}{a - 2} = \_\_\_\_\_\_, \ a \neq \_\_\_\_$ — $-a, \ a \neq 2$

57 $\frac{(-5x - 5)(2 - 2x)}{10x + 10} = \_\_\_\_\_\_, \ x \neq \_\_\_\_$ — $x - 1, \ x \neq -1$

In certain applications of mathematics, the number represented by the fraction $\frac{a}{b}$ is called the ratio of a to b. We shall sometimes speak of the ratio when we mean the symbol which indicates the quotient, provided there is no confusion in the meaning.

58 $\frac{x}{y}$ may be read "the ratio of ______ to ______". x to y

If the ratio of sophomores to freshmen is $\frac{5}{7}$, then

59 there are ______ sophomores to every 7 freshmen. 5

If the ratio of girls to boys is $\frac{3}{4}$, then there are 3

60 ______ to every 4 ______. girls, boys

In a certain college the ratio of faculty to students is $\frac{2}{19}$.

61 For every $\underset{\text{(how many)}}{\underline{\qquad\qquad}}$ faculty members there are 2

62 $\underset{\text{(how many)}}{\underline{\qquad\qquad}}$ students? 19

If there are "f" faculty members and 1197 students

63 then the ratio of faculty to students is $\frac{\square}{1197}$. $\frac{f}{1197}$

Since the faculty-student ratio is $\frac{2}{19}$, we know that

64 "$\frac{2}{19}$" and "$\frac{\square}{1197}$" both name the same number. $\frac{f}{1197}$

65 Hence, $\frac{f}{1197} = \frac{2}{\square}$. $\frac{2}{19}$

66 And $f =$ ______. 126

There are 126 faculty members.

The profits from a student assembly are to be given to the honor society and the mathematics club in the ratio of $\frac{2}{3}$ with the mathematics club receiving the larger amount, which is $387.

If h is the amount the honor society will receive, then an open sentence for the problem is

67 $\frac{\square}{\square} = \frac{2}{3}$. $\frac{h}{387} = \frac{2}{3}$

68 The honor society will receive $______. $258

13-2. Division of Fractions

For simplifying an indicated product of two or more fractions, a key property was the theorem which may be stated:

> For any real numbers $a, b, c, d,$ if $b \neq 0$ and $a \neq 0$, then
> $$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

In this section we shall see that an indicated quotient of two or more fractions may also be simplified into a phrase which will contain at most one indicated quotient.

Item	Frame	Answer
	To illustrate, let us consider $\dfrac{\frac{5}{3}}{\frac{7}{2}}$.	
1	This is an indicated _______ of two fractions.	quotient
2	It is also a fraction whose numerator is $\frac{5}{3}$ and whose denominator is _______.	$\frac{7}{2}$
3	The least common multiple of the denominators 2 and 3 is _______.	6
	Since $\frac{a}{b} = \frac{ak}{bk}$ if $b \neq 0$ and $k \neq 0$,	
4	$\dfrac{\frac{5}{3}}{\frac{7}{2}} = \dfrac{\frac{5}{3}\cdot 6}{\frac{7}{2}\cdot \square}$ _______	$\dfrac{\frac{5}{3}\cdot 6}{\frac{7}{2}\cdot 6}$
5	$= \dfrac{5\cdot\square}{7\cdot\square} = \dfrac{5\cdot 2}{3\cdot 7}$ _______	$\dfrac{5\cdot 2}{7\cdot 3}$
6	$= \frac{5}{3}\cdot$ _______ $= \frac{10}{21}$	$\frac{5}{3}\cdot\frac{2}{7}$
7	Since $\dfrac{\frac{5}{3}}{\frac{7}{2}}$ means $\frac{5}{3}\div\frac{7}{2}$, and from Items 4 and 6 we have $\dfrac{\frac{5}{3}}{\frac{7}{2}} = \frac{5}{3}\cdot\frac{2}{7}$, we see that $\frac{5}{3}\div\frac{7}{2} = \frac{5}{3}\cdot$ _______.	$\frac{5}{3}\div\frac{7}{2} = \frac{5}{3}\cdot\frac{2}{7}$

Item 7 suggests that an indicated quotient of two fractions may be expressed quite readily as an indicated product of two fractions. Moreover, the above procedure suggests a method for proving

<u>Theorem</u> <u>13-2</u>. For any real numbers a, b, c, and d, if $b \neq 0$, $c \neq 0$, and $d \neq 0$, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}.$$

To prove this theorem, we note that if $k \neq 0$,

8 $\frac{x}{y} = \frac{xk}{\square}$, $y \neq 0$. $\qquad \frac{x}{y} = \frac{xk}{yk}$

9 But $\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{\square}{\square}}$ $\qquad \frac{\frac{a}{b}}{\frac{c}{d}}$

10 $= \frac{\frac{a}{b} \cdot k}{\frac{c}{d} \cdot \square}$ $\qquad \frac{\frac{a}{b} \cdot k}{\frac{c}{d} \cdot k}$

If k is bd, we have

$$\frac{\frac{a}{b} \cdot k}{\frac{c}{d} \cdot k} = \frac{\frac{a}{b} \cdot bd}{\frac{c}{d} \cdot bd} = \frac{ad(\frac{b}{b})}{cb(\frac{d}{d})}$$

$$= \frac{ad}{cb} = \frac{ad}{bc}$$

11 $= \frac{a}{b} \cdot \underline{\qquad}$. $\qquad \frac{a}{b} \cdot \frac{d}{c}$

So $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$.

Notice that in the step before Item 11 we have the theorem in the handy form $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$.

Use the theorem to simplify each of the following:

	Exercise	Answer
12	$\frac{3}{8} \div \frac{1}{4} = \frac{3}{8} \cdot$ ______	$\frac{3}{8} \cdot \frac{4}{1} = \frac{3}{2}$
13	$\frac{5}{6} \div (-\frac{7}{3}) =$ ______	$-\frac{5}{14}$
14	$\frac{\frac{a^2}{2}}{\frac{a}{4}} =$ ______, $a \neq$ ______	$2a, \quad a \neq 0$
15	$\frac{\frac{a-3}{5}}{\frac{8}{a+1}} = \frac{(\quad)(\quad)}{\square}$, $a \neq -$	$\frac{(a-3)(a+1)}{40}$
16	$\frac{a-b}{2} \div \frac{a-b}{-4} =$ ______, $a \neq$ ______	$-2, \quad a \neq b$

Use the least common multiple of the denominators to simplify each of the following:

	Exercise	Answer
17	$\frac{\frac{3}{8}}{\frac{1}{4}} = \frac{\frac{3}{8} \cdot \square}{\frac{1}{4} \cdot \square} =$ ______	$\frac{\frac{3}{8} \cdot 8}{\frac{1}{4} \cdot 8} = \frac{3}{2}$
18	$\frac{\frac{5yz}{24x}}{\frac{25xz}{9y}} = \frac{\frac{5yz}{24x}(\quad)}{\frac{25xz}{9y}(\quad)}$	$\frac{\frac{5yz}{24x}(72xy)}{\frac{25xz}{9y}(72xy)}$
19	$=$ ______, $x \neq$ ______, $y \neq 0$, $z \neq 0$	$\frac{3y^2}{40x^2}, \quad x \neq 0$

Expressing an indicated quotient of two fractions as an indicated product of two fractions was very useful to us because we have already learned how to find the product of two fractions. Recall that we did something of this nature before in our definition of division of two real numbers.

57

For real numbers a and b, $(b \neq 0)$, $a \div b$

20 means a multiplied by the ______ of b. — reciprocal

This suggests another way of looking at the indicated quotient $\frac{a}{b} \div \frac{c}{d}$.

First, let's recall that the reciprocal of $\frac{1}{3}$

21 is ______. — 3 or $\frac{3}{1}$

22 The reciprocal of $\frac{3}{4}$ is ______. — $\frac{4}{3}$

23 The reciprocal of $\frac{c}{d}$ is ______, if $c \neq$ ___, $d \neq$ ___. — $\frac{d}{c}$, if $c \neq 0$, $d \neq 0$

By the definition of division,

24 $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b}\left(\frac{1}{\square}\right) = \frac{a}{b}\cdot\square = \frac{ad}{bc}$ — $\frac{a}{b}\left(\frac{1}{\frac{c}{d}}\right) = \frac{a}{b}\cdot\frac{d}{c}$

Thus we see that this agrees with the statement of Theorem 13-2.

Another method of simplifying an indicated quotient of two fractions makes use of the reciprocal and the special case of division, $\frac{x}{1}$, for a real number x.

25 $\frac{x}{1} =$ ______ — $\frac{x}{1} = x$

Let's consider the indicated quotient $\dfrac{\frac{5}{3}}{\frac{7}{2}}$.

26 The reciprocal of $\frac{7}{2}$ is ______, — $\frac{2}{7}$,

27 because $\frac{7}{2}\cdot\frac{2}{7} =$ ______. — 1

28 $\dfrac{\frac{5}{3}}{\frac{7}{2}} = \dfrac{\frac{5}{3}\cdot\frac{2}{7}}{\frac{7}{2}\cdot\square} = \dfrac{\frac{5}{3}\cdot\frac{2}{7}}{\square}$ — $\dfrac{\frac{5}{3}\cdot\frac{2}{7}}{\frac{7}{2}\cdot\frac{2}{7}} = \dfrac{\frac{5}{3}\cdot\frac{2}{7}}{1}$

29 $=$ ______ $= \frac{10}{21}$ — $\frac{5}{3}\cdot\frac{2}{7}$

Simplify each of the following using the method shown in Items 28 and 29.

30. $\dfrac{\frac{2}{3}}{-\frac{3}{4}}$ ______ — $-\frac{8}{9}$

31. $\dfrac{\frac{2}{x-5}}{\frac{3}{x-5}}$ ______, $x \neq$ ______ — $\frac{2}{3}$, $x \neq 5$

32. $\dfrac{\frac{2(1-a)}{a}}{\frac{6b}{a}}$ ______, $a \neq$ ______, $b \neq$ ______ — $\frac{1-a}{3b}$, $a \neq 0$, $b \neq 0$

33. $\dfrac{\frac{a}{b}}{\frac{c}{d}}$ ______, $b \neq 0$, $c \neq 0$, $d \neq 0$ — $\frac{ad}{bc}$

The following exercises will provide you with further practice in simplifying products or quotients of fractions. State the restrictions on the domains of the variables whenever it is necessary. The answers are on page ii.

34. $\frac{x+3}{x} \cdot \frac{x-3}{x}$

35. $\frac{2x-7}{7x-2} \div \frac{2x}{7x}$

36. $\frac{2x-7}{7x-2} \cdot \frac{6}{21-6x}$

37. $\dfrac{\frac{2x-7}{21}}{\frac{14-4x}{48y}}$

38. $\frac{45x(a+3)}{343yz} \div \frac{15(a+3)}{14(\cdot+\cdot)}$

5

13-3. Addition and Subtraction of Fractions

We have seen, in phrases which involve the product of several fractions, that we can always simplify the phrase to one which involves just one fraction. Likewise we can simplify phrases which involve the quotient of two fractions to one which involves just one fraction.

A phrase, however, may contain the sum of several fractions. Let us consider the addition of fractions. Since subtraction has been defined in terms of addition, we may consider subtraction at the same time. The phrase $\frac{x}{3} + \frac{y}{5}$ is an example of an indicated sum of two fractions. The phrase $\frac{x}{3} - \frac{y}{5}$ may be written $\frac{x}{3} + (-\frac{y}{5})$, which is also an indicated sum of two fractions.

1	The phrase "$\frac{x}{3} + \frac{y}{5}$" contains _______ indicated quotients.	two
2	x is divided by _______ and	3
3	y is divided by _______.	5
4	$\frac{x}{3} + \frac{y}{5}$ is not considered to be simplified since it contains more than _______ indicated _______.	one, quotient

	Let us consider an example from arithmetic.	
5	Simplify: $\frac{2}{7} + \frac{4}{7}$. This is an indicated _______ of two fractions.	sum
6	We already know that $\frac{2}{7} + \frac{4}{7} = \frac{\square}{7}$, and we can show that this agrees with the definitions of various operations and with various properties under these operations in the following manner.	$\frac{6}{7}$
7	$\frac{2}{7} = 2(\frac{1}{7})$ and $\frac{4}{7} = 4(\frac{1}{7})$ by the definition of _______.	division
	$\frac{2}{7} + \frac{4}{7} = 2(\frac{1}{7}) + 4(\frac{1}{7})$	
8	$= (2 + 4)(\frac{1}{7})$ by the _______ property.	distributive
	$= 6(\frac{1}{7})$	
9	$=$ _______ by the definition of division.	$\frac{6}{7}$

10	In a similar manner, we may show that for real numbers a, b, c, with $c \neq 0$, $\frac{a}{c} + \frac{b}{c} = \underline{\qquad}$. Try to construct the proof by yourself and compare your proof with the one on page ii.	$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$

The statement $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ agrees with the method we use to add two fractions in arithmetic when the denominators are alike. You will recall that when the denominators are different, we first rewrite each fraction so that the denominators are alike.

	For example, to simplify $\frac{3}{5} + \frac{2}{7}$, we note that	
11	$\frac{3}{5} = \frac{3 \cdot 7}{5 \cdot \square} = \frac{21}{35}$ since $\frac{a}{b} = \frac{ak}{bk}$ if $b \neq 0$ and $k \neq 0$.	$\frac{3}{5} = \frac{3 \cdot 7}{5 \cdot 7}$
12	Similarly, $\frac{2}{7} = \frac{2 \cdot \square}{7 \cdot \square} = \frac{\square}{35}$.	$\frac{2}{7} = \frac{2 \cdot 5}{7 \cdot 5} = \frac{10}{35}$
13	So $\frac{3}{5} + \frac{2}{7} = \frac{21}{\square} + \frac{10}{\square} = \underline{\qquad}$.	$\frac{21}{35} + \frac{10}{35} = \frac{31}{35}$
	We notice that each fraction is rewritten as a fraction whose denominator is 35, and that 35 is the	
14	least ______ ______ of 5 and 7.	common multiple

For examples such as $\frac{3}{5} + \frac{2}{7}$ it is easy to find a denominator in terms of which we can express each of the fractions. In this usage, we refer to the number 35 as the least common denominator of $\frac{3}{5}$ and $\frac{2}{7}$. In Chapter 4, we also referred to such a number as the least common multiple of the denominators; that is, 35 is the least common multiple of 5 and 7. Since $35 = 5 \cdot 7$, we can see that 35 is a multiple of 5, and 35 is a multiple of 7. Moreover, it is the smallest positive number that is a multiple both of 5 and of 7.

Definition. The least common multiple of two or more given integers is the smallest positive integer which is divisible by all of the given integers.

It is not hard to find the least common multiple of certain integers. For example, with very little experimenting, we can quickly find that 20 is the least common multiple of 4 and 10. Finding the least common multiple of 51 and 85 is not as easily done by inspection. For this purpose, we shall find that prime factorization will come in handy.

15	The prime factorization of 51 is ______.	$3 \cdot 17$
16	The prime factorization of 85 is ______.	$5 \cdot 17$
17	If k is an integer, then ____ k is a multiple of 51.	51k
18	Since 51k is a multiple of 51, $\underline{3 \cdot 17 \cdot k}$ is a multiple of 51.	$3 \cdot 17 \cdot k$
19	To be a multiple of 51, an integer must have at least the factors 3 and ____.	17
20	If r is an integer then 85r is a ______ of 85.	multiple
21	By the prime factorization of 85, the number 85r = ______.	$5 \cdot 17r$
22	To be a multiple of 85, an integer must have at least the factors ________.	5 and 17
23	A common multiple of 51 and 85 must have at least the factors ____, ____, and ____.	3, 5, 17
24	The least common multiple of 51 and 85 is ___ $\cdot$ ___ $\cdot$ ___ = 255.	$3 \cdot 5 \cdot 17$
25	$\frac{7}{51} - \frac{2}{85} = \frac{7}{3 \cdot 17} - \frac{2}{5 \cdot \square}$	$\frac{7}{3 \cdot 17} - \frac{2}{5 \cdot 17}$
26	$= \frac{7 \cdot \square}{3 \cdot 17 \cdot 5} + \frac{-2 \cdot \square}{5 \cdot 17 \cdot \square}$	$\frac{7 \cdot 5}{3 \cdot 17 \cdot 5} + \frac{-2 \cdot 3}{5 \cdot 17 \cdot 3}$
27	$= \frac{\square + \square}{3 \cdot 5 \cdot 17} = \frac{\square}{3 \cdot 5 \cdot 17}$	$\frac{35 + (-6)}{3 \cdot 5 \cdot 17} = \frac{29}{3 \cdot 5 \cdot 17}$

We can find the least common multiple of more than two integers in much the same way as we do for two integers. To find the least common multiple of 3, 8, and 70, we can first get the prime factorization of each.

28	$3 = 3$, $8 = 2 \cdot 2 \cdot 2$, and $70 =$ ______.	$2 \cdot 5 \cdot 7$

The common multiple of 3, 8, and 70 must have at least: 3 as a factor once, ____ as a factor three times, 5 as a factor once, and ____ as a factor ____. It is $2\cdot2\cdot2\cdot3\cdot5\cdot7$.	2 7 once

Simplify:	
$\frac{1}{3} + \frac{5}{8} - \frac{11}{70} = \frac{1(\quad)}{3(2\cdot2\cdot2\cdot5\cdot7)} + \frac{5(\quad)}{(2\cdot2\cdot2)(\quad)} - \frac{\square}{\square}$	See answer below.
$= \frac{1(2\cdot2\cdot2\cdot5\cdot7)}{3(2\cdot2\cdot2\cdot5\cdot7)} + \frac{5(3\cdot5\cdot7)}{(2\cdot2\cdot2)(3\cdot5\cdot7)} - \frac{11\cdot2\cdot2\cdot3}{(2\cdot5\cdot7)2\cdot2\cdot3}$	
33 $= \frac{280 + \square - \square}{2\cdot2\cdot2\cdot3\cdot5\cdot7}$	$\frac{280+525-132}{2\cdot2\cdot2\cdot3\cdot5\cdot7}$
34 $= \frac{\square}{840}$	$\frac{673}{840}$
[illegible] $= \frac{\square}{\square}$	$\frac{-3}{7\cdot13}$
$= \frac{\square}{2\cdot3\cdot7\cdot13} =$ ____	$\frac{-26}{2\cdot3\cdot7\cdot13} = -\frac{1}{21}$
37 $\frac{x+8}{10} + \frac{x-4}{4} = \frac{(x+8)2}{(2\cdot5)2} + \frac{(x-4)\square}{(2\cdot2)\square}$	$\frac{(x+8)2}{(2\cdot5)2} + \frac{(x-4)5}{(2\cdot2)5}$
$= \frac{(2x+16) + (5x-20)}{\square}$	$\frac{(2x+16)+(5x-20)}{20}$
$=$ ____	$\frac{7x-4}{20}$

40 We can rewrite $\frac{a}{b}$ as $\frac{\square}{bk}$ if $k \neq 0$.	$\frac{ak}{bk}$
Consider the example $\frac{2}{x} + \frac{3}{7x}$, $x \neq 0$. The first fraction $\frac{2}{x}$ can be written	
41 $\frac{2\cdot7}{x\cdot\square} = \frac{14}{7x}$.	$\frac{2\cdot7}{x\cdot7}$

42	$\frac{2}{x} + \frac{3}{7x}$ can be rewritten as $\frac{14}{\square} + \frac{3}{\square}$ ________ .	$\frac{14}{7x} + \frac{3}{7x}$
	Since the fractions in this last phrase have the	
43	same ______________ (numerator, denominator), we can make use of the	denominator
	equality,	
44	$\frac{a}{c} + \frac{b}{c} =$ ________ .	$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$
45	From this, $\frac{2}{x} + \frac{3}{7x} = \frac{14}{7x} + \frac{3}{7x} =$ ______ .	$\frac{17}{7x}$

We see that the same technique we used when the denominators are integers can be used when the denominators involve variables. (Of course we must be careful of the restrictions on the domain of the variables.) We rewrite the fractions with a common denominator; the denominators of the original fractions are factors of the common denominator.

	For example, if the denominators are x and $7x$, a	
46	common denominator is ______ because x is a	$7x$
47	factor of $7x$, and $7x$ is also a factor of ______ .	$7x$
	To simplify $\frac{4c}{3ab} + \frac{5b}{2a^2} - \frac{5}{12}$, we note that the denominators are $3ab$, $2a^2$, and 12.	
	These can be written as products:	
48	$3 \cdot a \cdot b$, $2 \cdot a \cdot a$, and $\_\cdot\_\cdot\_$.	$2 \cdot 2 \cdot 3$
	The common denominator must have at least 3 as	
49	a factor once, 2 as a factor ______, a as a	twice
50	factor ______, and ____ as a factor once.	twice, b
51	A common denominator is $2 \cdot 2 \cdot 3 \cdot$ ________ .	$2 \cdot 2 \cdot 3 \cdot a \cdot a \cdot b$
52	$\frac{4c}{3ab} + \frac{5b}{2a^2} - \frac{5}{12} = \frac{4c(\quad)}{3ab(4a)} + \frac{5b\square}{\square} - \frac{\square}{\square}$	See answer below.
	$= \frac{4c(4a)}{3ab(4a)} + \frac{5b(6b)}{2a^2(6b)} - \frac{5(a^2b)}{12(a^2b)}$	
53	$= \frac{16ac + 30b^2 - 5a^2b}{\square}$	$\frac{16ac+30b^2-5a^2b}{12a^2b}$
54	$= \frac{\square}{\square}$, $a \neq$, $b \neq$	$\frac{16ac+30b^2-5a^2b}{12a^2b}$, $a \neq 0$, $b \neq 0$

Let's try a few more examples.

55 $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} =$ ______. — $\frac{13}{12}$

56 $\frac{x}{4} - \frac{x}{2} =$ ______. — $-\frac{x}{4}$

57 $\frac{4}{a} + \frac{5}{2a} = \frac{4 \cdot 2}{a \cdot 2} + \frac{5}{2a} =$ ______, $a \neq$ ______ — $\frac{13}{2a}$, $a \neq 0$

58 $\frac{5}{x+2} + \frac{7}{3x+6} = \frac{5 \cdot \square}{(x+2)\square} + \frac{7}{3(\quad)} =$ ______, $x \neq$ ______ — $\frac{5 \cdot 3}{(x+2)3} + \frac{7}{3(x+2)}$ $= \frac{22}{3(x+2)}$, $x \neq -2$

59 $\frac{10}{x+2} + \frac{5}{2x-1} = \frac{10(2x-1)}{(x+2)(2x-1)} + \frac{5(\quad)}{(2x-1)(x+2)}$ — $\frac{5(x+2)}{(2x-1)(x+2)}$

60 $=$ ______, $x \neq$ ______, $x \neq$ ______ — $= \frac{25x}{(x+2)(2x-1)}$ $x \neq -2$, $x \neq \frac{1}{2}$

61 $\frac{5}{x+2} - \frac{7}{2x-1} = \frac{5}{x+2} + \frac{-7}{2x-1} =$ ______, $x \neq$ ______, $x \neq$ ______ — $\frac{3x-19}{(x+2)(2x-1)}$ $x \neq -2$, $x \neq \frac{1}{2}$

62 $\frac{5}{2x-1} - \frac{7}{x+2} =$ ______, $x \neq -2$, $x \neq \frac{1}{2}$ — $\frac{17-9x}{(x+2)(2x-1)}$

We can use the ideas of this section to find the truth sets of the following. (Solution of other open sentences involving fractions will be discussed in further detail in Section 19-2.)

To solve $\frac{2x}{3} + \frac{x}{4} = 5$, we may simplify the expression on the left:

63 $\frac{\square}{12} = 5$ — $\frac{11x}{12} = 5$

64 Since $\frac{11x}{12} = \frac{11}{12}x$, we have $\square x = 5$. — $\frac{11}{12}x = 5$

65 Let us multiply by the reciprocal of $\frac{11}{12}$ on ______ side(s) of the equation. — both

(the left, the right, both)

If $\frac{11}{12}x = 5$ is true for some x, then

66 $\frac{12}{11} \cdot \frac{11}{12}x = \square \cdot 5$ is true for the same x. $\frac{12}{11}$

From this, $x = \frac{60}{11}$, and the truth set of $\frac{2x}{3} + \frac{x}{4} = 5$

67 is ______. $\{\frac{60}{11}\}$

Notice that we could have solved this in another

68 way. Since ______ is the least common multiple of 3 and 4, we can first multiply both sides by 12. 12

$$\frac{2x}{3} + \frac{x}{4} = 5$$

$$12\left(\frac{2x}{3} + \frac{x}{4}\right) = 12 \cdot 5$$

69 $8x + \_\_\_ = 60$ $3x$

70 $\_\_\_\_\_\_ = 60$ $11x = 60$

71 $x = \_\_\_$ $x = \frac{60}{11}$

Solve:

72 $\frac{7}{9}x = \frac{1}{3}x + 8$ Solution set: ______ $\{18\}$

73 $\frac{1}{4}y + 3 = \frac{1}{2}y$ Solution set: ______ $\{12\}$

74 $\frac{1}{x} + \frac{2}{x} = 6, \quad x \neq 0$ Solution set: ______ $\{\frac{1}{2}\}$

Find the solution set of

75 $3|w| + 8 = \frac{1}{2}|w| + \frac{41}{2}$ $\{5, -5\}$

76 $-\frac{3}{7} + |x - 3| < \frac{22}{14}$ set of all real numbers between 1 and 5

If you missed either of these see the complete solution on page iii.

Solve each of the following by first writing the appropriate open sentence and then finding its truth set. In many cases the open sentences you write will involve fractions. When you have solved all of these problems check your work with that shown on page iii.

77. The sum of two numbers is 240, and one number is $\frac{3}{5}$ times the other. What are the two numbers?

78. The numerator of the fraction $\frac{4}{7}$ is increased by an amount x. The value of the resulting fraction is $\frac{27}{21}$. By what amount was the numerator increased?

79. Joe is $\frac{1}{3}$ as old as his father. In 12 years he will be $\frac{1}{2}$ as old as his father then is. How old is Joe? How old is his father?

80. The sum of two positive integers is 7, and their difference is 3. What are the integers? What number is the result if the reciprocal of the smaller is decreased by the reciprocal of the larger?

81. In a shipment of 800 radios, $\frac{1}{20}$ of the radios were defective. What is the ratio of defective to non-defective radios in the shipment?

82	If it takes Joe 7 days to paint his house, what part of the job will he do in one day? What part in d days?	$\frac{1}{7}$, $\frac{d}{7}$
83	If it takes Bob 8 days to paint Joe's house, what part of the job would he do in one day? What part in d days?	$\frac{1}{8}$, $\frac{d}{8}$
84	If Bob and Joe work together, what portion of the job would they do in one day?	$\frac{1}{7} + \frac{1}{8}$
85	What portion in d days?	$\frac{d}{7} + \frac{d}{8}$

The following items refer to the questions asked in Items 82-85. The answers are discussed on page iv.

The open sentence suggested by the problem is $\frac{d}{7} + \frac{d}{8} = 1$.

86. Solve the equation $\frac{d}{7} + \frac{d}{8} = 1$. What does d represent?

87. What portion of the painting will Joe and Bob, working together, do in one day?

The following exercises will provide you with further practice in simplifying quotients and sums of fractions. State the domains of the variables whenever it is necessary.

88	$\dfrac{\frac{2}{3}+\frac{1}{6}}{\frac{5}{6}} = \dfrac{(\frac{2}{3}+\frac{1}{6})\cdot\square}{\frac{5}{6}\cdot 6}$ = ________	$\dfrac{(\frac{2}{3}+\frac{1}{6})\cdot 6}{\frac{5}{6}\cdot 6} = 1$
89	$\dfrac{\frac{a-b}{2}}{\frac{a-b}{-4}}$ = ________ , $a \neq$	$-2,\ a \neq b$
90	$\dfrac{\frac{1}{3}+\frac{1}{4}}{\frac{1}{3}-\frac{1}{4}}$ = ________	7
91	$\dfrac{3x}{4y}+\dfrac{2}{3x}$ = ________ , $x \neq$ ____ , $y \neq$ ____	$\dfrac{9x^2+8y}{12xy}$ $x \neq 0,\ y \neq 0$
92	$\dfrac{\frac{x+5}{9}}{\frac{3}{x+2}}$ = ________	$\dfrac{(x+5)(x+2)}{27},\ x \neq -2$
93	$\dfrac{a}{b}+\dfrac{c}{d}$ = ________	$\dfrac{ad+bc}{bd},\ b \neq 0, d \neq 0$

13-4. Summary and Review

Theorem 13-1: For any real numbers a, b, c, d, if $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}.$$

Theorem 13-2: For any real numbers a, b, c, d, if $b \neq 0$, $c \neq 0$, and $d \neq 0$, then

$$\frac{a}{b}\div\frac{c}{d} = \frac{a}{b}\cdot\frac{d}{c} = \frac{ad}{bc}.$$

For any real numbers $a, b, c,$ if $c \neq 0$, then

$$\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}.$$

For any real numbers $a, b, c, d,$ if $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}.$$

For any real numbers $a, b,$ if $b \neq 0$,

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}.$$

The least common multiple of two or more given integers is the smallest possible integer which is divisible by all of the given integers.

When we find the common name of (simplify) an expression, we try to keep to the following conventions:

1. A common name contains no indicated division if it can be avoided.
2. If a common name must contain an indicated division, then the resulting expression should be written in "lowest terms".
3. We prefer writing $-\frac{a}{b}$ to either of the forms $\frac{-a}{b}$ or $\frac{a}{-b}$.

Review

The answers to the following review problems are on page iv.

1. Simplify each of the following expressions; be sure to state the domain of the variable whenever necessary.

(a) $\frac{1}{4} - \frac{1}{3} + \frac{1}{6}$

(b) $\frac{27}{35} - \frac{19}{21}$

(c) $\frac{5m}{12} - \frac{7m}{18} - \frac{1}{2}$

(d) $\frac{a}{44} + \frac{2a}{33} + \frac{3a}{22}$

(e) $\frac{5m}{12} + \frac{7m}{18}$

(f) $\dfrac{5}{\frac{1}{8} + \frac{1}{12}}$

(g) $\dfrac{5 - \frac{1}{a}}{3a - \frac{3}{5}}$

(h) $\frac{a + 7}{2a - 5} \cdot \frac{-7}{21 + 3a}$

(i) $\frac{7}{2(x - 2)} - \frac{5}{3(2x + 5)}$

2. Find the truth set of $\frac{x}{3} + \frac{5}{12} = 12 + \frac{1}{4}x$.

[Hint: if x is a number that makes the above sentence true, then x is a number that makes $\frac{x}{3} - \frac{1}{4}x = 12 - \frac{5}{12}$ true.]

3. Kevin has five hours at his disposal. How far can he ride his bicycle into the surrounding hills at the rate of 12 miles per hour and return by retracing his route at the rate of 8 miles per hour?

Chapter 14

EXPONENTS

14-1. Introduction to Exponents

For the prime factorization of the positive integer 288 we have written

$$288 = (2)(2)(2)(2)(2)(3)(3).$$

This notation is inconvenient and clumsy because it is so lengthy. We could avoid this form if there were a more compact way to express the product of a number of repeated factors.

	You already know that (3)(3) may be written as 3^2.	
	Similarly,	
1	$(2)(2) = \underline{2^{\square}}$	2^2
2	$7 \cdot 7 = \underline{\square^2}$	7^2
3	$(13{,}849)(13{,}849) =$ ______	$13{,}849^2$
4	$11^2 = \underline{(\quad)(\quad)}$	(11)(11)
5	$6^2 = \underline{\quad\cdot\quad}$	$6 \cdot 6$
6	$y \cdot y =$ ______	y^2
	3^2 is read as "3 squared".	
7	"17 squared" means $\underline{(17)(\ \ )}$.	(17)(17)
8	In the numeral 5^2, the "2" indicates that we are using the number 5 ________ (how many) times as a factor.	two
9	If the length of the side of a square is s units, then the area of the square, in <u>square</u> <u>units</u>, is s^2, which is read, "s____".	squared
10	If the length of the edge of a cube is e units, then the volume of the cube, in <u>cubic</u> units, is $\underline{(e)(\ )(\ )}$, or "e cubed".	(e)(e)(e)
11	An appropriate symbol for (e)(e)(e) is e^3, the "3" indicating that e is used as a factor ________ (how many) times.	three

12	$5^3 = \underline{(5)(\ )(\ )}$.	(5)(5)(5)
13	$2^3 = $ ______.	(2)(2)(2)
14	$(7)(7)(7) = \underline{7^{\square}}$.	7^3
15	$(10)(10)(10) = \underline{\square^3}$.	10^3

It is natural to extend the idea used in writing 2^2, 2^3, to such products as $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$. In this expression, 2 is used as a factor five times. We agree to write

$$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5.$$

16	$2^4 = \underline{2 \cdot 2 \cdot \quad \cdot \quad}$.	$2 \cdot 2 \cdot 2 \cdot 2$
17	2^7 means the product formed by using 2 ______ (how many) times as a factor.	seven
	Find the common name:	
	$2^3 = 8$	
18	$2^4 = $ ______	16
19	$2^5 = $ ______	32
20	$3^4 = $ ______	81
21	$5^2 = $ ______	25
22	$7^3 = $ ______	343
23	$1^{19} = $ ______	1

24	$288 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$ $= \underline{(2^{\square})(3^{\square})}$	$(2^5)(3^2)$
	The prime factorization of 1000 is $2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5$.	
25	Using the more compact notation, $1000 = \underline{(2^3)(\ \ )}$.	$(2^3)(5^3)$
	Find the prime factorization of each of the following, using exponents whenever appropriate.	
26	$64 = $ ______	2^6
27	$60 = $ ______	$(2^2)(3)(5)$
28	$80 = $ ______	$(2^4)(5)$
29	$81 = $ ______	3^4

30	$41 =$ ______	41 [prime]
31	$243 =$ ______	3^5
32	$432 =$ ______	$(2^4)(3^3)$
33	$768 =$ ______	$(2^8)(3)$
34	$242 =$ ______	$(2)(11^2)$
35	$1248 =$ ______	$(2^5)(3)(13)$

In general, if a is to be used n times as a factor, we shall write a^n. Thus,

$$a^n = \underbrace{(a)(a)(a)\ldots(a)}_{n \text{ factors}}$$

We need some language to use in describing the numbers involved in the expression a^n. The "a", which indicates the number to be used as a factor, is called the <u>base</u>. The "n", which indicates how many times the factor is to be used, is called the <u>exponent</u>. We sometimes refer to a^n as "a to the n^{th} power", or simply as "a to the n^{th}."

36	In the expression 7^6, the base is ______	7
37	and the exponent is ______.	6
38	We read 7^6 as "___ to the ___".	7 to the 6^{th} power
39	In the expression y^m, y is the ______ and	base
40	m is the ______.	exponent
41	y^m indicates that ______ should be used as a	y
42	factor ______ times.	m

43	$(-\frac{2}{3})^3 = (\ \ )(\ \ )(\ \ )$	$(-\frac{2}{3})(-\frac{2}{3})(-\frac{2}{3})$
44	$=$ ______	$-\frac{8}{27}$
45	$(-4)^4 = (-4)$ ______	$(-4)(-4)(-4)(-4)$
46	$=$ ______	256
47	$(\sqrt{2})^2 = (\ \ )$ ______	$(\sqrt{2})(\sqrt{2})$
48	$=$ ______	2

	Find a common name for each of the following:	
49	$(5)^3 =$ ______	125
50	$(\frac{1}{2})^2 =$ ______	$\frac{1}{4}$
51	$(-3)^4 =$ ______	81
52	$(-2)^3 =$ ______	-8
53	$(-\frac{4}{3})^2 =$ ______	$\frac{16}{9}$
54	$(\frac{1}{\sqrt{3}})^4 =$ ______	$\frac{1}{9}$

We have not yet mentioned a^1. We shall define $a^1 = a$. Since a is a simpler numeral than a^1, we shall usually write a in place of a^1.

	$7^1 = 7$	
55	$3^1 =$ ______	3
56	$(-\sqrt{2})^1 =$ ______	$-\sqrt{2}$
57	$\underline{(-\frac{1}{3})^{\square}} = -\frac{1}{3}$	$(-\frac{1}{3})^1$

Examine the next items carefully. Can you find a general pattern?

	$2^3 \cdot 2^2 = (2 \cdot 2 \cdot 2)(2 \cdot 2)$	
	$= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$	
58	$= \underline{2^{\square}}$	2^5
59	$3^4 \cdot 3^2 = (3 \cdot 3 \cdot 3 \cdot 3)\underline{(\ \ \cdot\ \ )}$	$(3 \cdot 3)$
60	$= \underline{3^{\square}}$	3^6
61	$a^2 \cdot a^5 = \underline{a^{\square}}$	a^7

a^m is the product formed by using a as a factor m times, a a^n is the product formed by using a as a factor n times. In the product $a^m \cdot a^n$, a is used as a factor $m + n$ times in all.

$$a^m \cdot a^n = \overbrace{\underbrace{(a)(a)(a)\ldots(a)}_{m\ \text{factors}}\ \underbrace{(a)(a)(a)\ldots(a)}_{n\ \text{factors}}}^{m+n\ \text{factors}} = a^{m+n}$$

m factors and n factors

Write simpler names for the following:

Example: $(9x^2)(3x^4) = 27x^6$

62	$m^3m^{11} = \underline{\qquad}$	m^{14}
63	$(x^3)(x^9) = \underline{\qquad}$	x^{12}
64	$(3x)(x) = \underline{\qquad}$	$3x^2$
65	$(3x^2)(3x)^2 = \underline{(3x^2)(3x)(\quad)}$	$3x$
	$= (3)(3)(3)(x^2)(x)(x)$	
66	$= \underline{3^{\square}x^{\square}}$ or $\underline{\qquad x^4}$	3^3x^4, $27x^4$
67	$(2x)(2x^3) = \underline{\qquad}$ [remember $x = x^1$]	$4x^4$
68	$(2x)(2^3x^3) = \underline{2^{\square}x^{\square}} = \underline{\qquad}$	2^4x^4, $16x^4$
69	$(16a^2)(32a^8) = \underline{2^{\square}a^{\square}}$ or $512a^{10}$	2^9a^{10}
70	$(x^{2a})(x^a) = \underline{\qquad}$ [Hint: $2a + a = 3a$]	x^{3a}
71	$3^4 \cdot 3^2 = \underline{\qquad}$ [Leave in exponent term]	3^6
72	$3^4 \cdot 2^3 = \underline{\qquad}$ Careful!	$3^4 \cdot 2^3$ or 648
73	$2^5\cdot3^2\cdot5\cdot2^2\cdot3^3\cdot5^2 = (2^5\cdot2^2)(3^2\cdot3^3)(5\cdot5^2) = \underline{2^{\square}\cdot 3^{\square}\cdot 5^{\square}}$	$2^7\cdot3^5\cdot5^3$
	$(3a^2b^3)(3^2ab^2) = (3\cdot3^2)(a^2\cdot a)(b^3\cdot b^2)$	
74	$= \underline{3^{\square}a^{\square}b^{\square}}$	$3^3a^3b^5$
75	$= \underline{\qquad}$	$27a^3b^5$
76	$(3xy^3)(2x^3y^4)(xy) = (3\cdot2)(x\cdot x^3\cdot x)\underline{(\qquad)}$	$(y^3\cdot y^4\cdot y)$
77	$= \underline{6x^{\square}y^{\square}}$	$6x^5y^8$
78	$(5c^4d^5)(4c^3x^2d) = \underline{20c^{\square}d^{\square}x^{\square}}$	$20c^7d^6x^2$
79	$(4am^7)(a^3m) = \underline{\qquad}$	$4a^4m^8$

At a glance $-x^2$ and $(-x)^2$ look alike and often cause difficulty unless we distinguish carefully between the two.

Frame		Answer
80	$-x^2 = \underline{-(x)(\quad)}$, also	$-(x)(x)$
81	$(-1)(x)(x) = \underline{-x^{\square}}$	$-x^2$
82	$-(x)(x) =$ ______	$-x^2$
83	$(-x)^2 = \underline{(\quad)(\quad)}$	$(-x)(-x)$
	$= (-1)(-1)(x)(x)$	
84	$=$ ______	x^2
85	$(-2)(x)(x) =$ ______	$-2x^2$
86	$(-2x)^2 = \underline{(\qquad)(\qquad)}$	$(-2x)(-2x)$
87	$= \underline{(-2)(\quad)(x)(\quad)}$	$(-2)(-2)(x)(x)$
	$= 4x^2$	
88	$(-3a)^2 =$ ______	$9a^2$
89	$(-4a)^3 =$ ______	$-64a^3$
90	$-4a^3 =$ ______	$-4a^3$

Frequently we wish to rewrite certain expressions using the distributive property:

$$2x^2(2^3x^2 + 2x) = 2x^2(2^3x^2) + (2x^2)(2x)$$
$$= 2^4x^4 + 2^2x^3$$
$$= 16x^4 + 4x^3$$

For each of the following write another name which does not contain parentheses.

Frame		Answer
91	$y^3(y^2 + 2) =$ ______	$y^5 + 2y^3$
92	$x^2(2x^3 + x^2) =$ ______	$2x^5 + x^4$
93	$2x^3(2x^2 - 4x^3) =$ ______	$2^2x^5 - 2^3x^6$, or $4x^5 - 8x^6$
94	$-3a^4(3^2a^3 - 3^3a) =$ ______	$-3^3a^7 + 3^4a^5$, or $-27a^7 + 81a^5$

Frame		Answer
*95	$(a^2 + 2a^3)(a - a^2) = a^2(a - a^2) + 2a^3(a - a^2)$	
	$=$ ______	$-2a^5 + a^4 + a^3$

*96 If we restrict a to the set of positive integers, then a^n defines a binary operation in this set. Is the operation commutative? That is, does a^n equal n^2 ?

[A] yes [B] no

A single example shows that the operation is not commutative. $2^3 = 8$, but $3^2 = 9$. Thus, the correct choice is [B].

*97 Again referring to the set of positive integers, does a^n define an associative operation in the sense that $(a^n)^m$ and $a^{(n^m)}$ name the same number?

[A] yes [B] no

$(2^2)^3 = 2^2 \cdot 2^2 \cdot 2^2 = 2^6$, but $2^{(2^3)} = 2^{(2\cdot 2\cdot 2)} = 2^8$. Therefore, the operation is not associative and the correct choice is [B]. You might like to reread Section 4-3, Items *68-*75. In those items we asked these same questions, using different notation.

14-2. Positive Integers as Exponents

Let us examine the fraction $\frac{a^5}{a^3}$, where $a \neq 0$. Can we find a simpler name?

$a^5 = a\cdot a\cdot a\cdot a\cdot a$

1 Similarly, $a^3 =$ ______ $a\cdot a\cdot a$

$\frac{a^5}{a^3} = \frac{a\cdot a\cdot a\cdot a\cdot a}{a\cdot a\cdot a}$

2 $= \left(\frac{a\cdot a\cdot a}{a\cdot a\cdot a}\right)(\ \cdot\ )$ ______ $\left(\frac{a\cdot a\cdot a}{a\cdot a\cdot a}\right)(a\cdot a)$

3 $= (\ )(a\cdot a)$ ______ $1(a\cdot a)$

Hence,

4 $\frac{a^5}{a^3} =$ ______ $\frac{a^5}{a^3} = a^2$

Find a simpler name for each of the following, where none of the variables has the value 0.

5 $\frac{x^5}{x^2} = x \cdot x \cdot x(\frac{x \cdot x}{x \cdot x}) = x^3 \cdot 1 =$ ______ x^3

6 $\frac{m^7}{m^5} = (\frac{m \cdot m \cdot m \cdot m \cdot m}{m \cdot m \cdot m \cdot m \cdot m}) \cdot m \cdot m =$ ______ m^2

7 $\frac{y^4}{y} =$ ______. Remember $y = y^1$ y^3

In each of the above items, the exponent in the

8 numerator is ______ than the exponent in the denominator. (greater, less) greater

In the example, $\frac{x^5}{x^2} = x^3$, the exponents are 5, 2 and 3.

9 $\frac{m^7}{m^5} = m^2$, the exponents are 7, 5 and ______. 7, 5 and 2

10 In $\frac{y^4}{y} = y^3$, the exponents are ___, 1 and 3. 4, 1 and 3

Did you observe a basic pattern which could be used in each of these examples without tediously writing each factor?

11 $\frac{x^5}{x^2} = x^{5-\square} = x^3$ where $x \neq 0$. x^{5-2}

12 $\frac{m^7}{m^5} = m^{\square-\square} = m^2$ where $m \neq 0$. m^{7-5}

13 $\frac{y^4}{y} = y^{\square-\square} =$ ______ where $y \neq 0$. $y^{4-1} = y^3$

It appears that for any real number a, different from 0, and for any positive integers m and n with m greater than n.

14 $\frac{a^m}{a^n} = a^{\square}$ $\frac{a^m}{a^n} = a^{m-n}$

The proof follows.

15 $\frac{a^m}{a^n} = \frac{a^n \cdot a^{\square}}{a^n}$ (Hint: $a^7 = a^3 \cdot a^{7-3}$) $m-n$

$= (\frac{a^n}{a^n})a^{m-n}$

16 $= \underline{\quad} \cdot a^{m-n}$ $1 \cdot a^{m-n}$

$= a^{m-n}$

If $a \neq 0$ and $m = n$, then $\frac{a^m}{a^n} = \frac{a^m}{a^m} = 1$.

17 $\frac{a^2}{a^2} = \underline{\qquad}$ 1

18 $\frac{z^3}{z^3}$, $\frac{b^5}{b^5}$, and $\frac{y^7}{y^7}$ are all names for _______, if the variables are all different from 0. 1

In the previous items the exponent in the numerator was greater than or equal to the exponent in the denominator. What if the exponent in the denominator is the greater?

$\frac{b^2}{b^5} = \frac{b \cdot b}{b \cdot b \cdot b \cdot b \cdot b}$

$= \frac{b \cdot b \cdot 1}{b \cdot b \cdot b \cdot b \cdot b}$

19 $= (\frac{b \cdot b}{b \cdot b}) \cdot \frac{\square}{b \cdot b \cdot b}$ 1

20 $= \underline{\qquad}$ where $b \neq 0$. $\frac{1}{b^3}$

$\frac{x}{x^3} = \frac{x}{x \cdot x \cdot x}$ $(x = x^1)$

$= \frac{x \cdot 1}{x \cdot x \cdot x}$

21 $= (\quad) \cdot \frac{1}{x \cdot x}$ $\frac{x}{x}$

22 $= \underline{\qquad}$ where $x \neq 0$. $\frac{1}{x^2}$

23	$\frac{m^4}{m^6} = \frac{1}{m^{6-\square}} =$ ______ where $m \neq 0$.	$\frac{1}{m^{6-4}}$, $\frac{1}{m^2}$
24	$\frac{c^{13}}{c^{17}} = \frac{1}{c^{\square-\square}} =$ ______ where $c \neq 0$.	$\frac{1}{c^{17-13}}$, $\frac{1}{c^4}$
25	$\frac{3^4}{3^6} = \frac{1}{3^{\square}} = \frac{1}{9}$	$\frac{1}{3^2}$
	When $n > m$ and $a \neq 0$ it appears that	
26	$\frac{a^m}{a^n} = \frac{1}{a^{\square-\square}}$	$\frac{1}{a^{n-m}}$

On a separate sheet of paper, try to prove

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}},$$

where $n > m$, $a \neq 0$.

When you finish, compare your proof with the following.

$$\frac{a^m}{a^n} = \frac{a^m}{a^m \cdot a^{n-m}} \qquad (n = m + (n-m))$$

$$= \frac{a^m}{a^m} \cdot \frac{1}{a^{n-m}}$$

$$= 1 \cdot \frac{1}{a^{n-m}}$$

$$= \frac{1}{a^{n-m}}$$

To summarize what we have shown about $\frac{a^m}{a^n}$:

If $a \neq 0$, if m and n are positive integers, and if

(a) $m > n$, then $\frac{a^m}{a^n} = a^{m-n}$ (Example: $\frac{6^5}{6^3} = 6^2$)

(b) $m = n$, then $\frac{a^m}{a^n} = \frac{a^m}{a^m} = 1$ (Example: $\frac{5^2}{5^2} = 1$)

(c) $m < n$, then $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$ (Example: $\frac{6^5}{6^9} = \frac{1}{6^4}$)

In each of the following write a simpler name for the fraction.

27 $\frac{x^9}{x^3} =$ ______, $(x \neq 0)$ — x^6

28 $\frac{2a^3}{a^3} =$ ______, $(a \neq 0)$ — 2

29 $\frac{y^{10}}{y^{13}} =$ ______, $(y \neq 0)$ — $\frac{1}{y^3}$

30 $\frac{2a^3}{a^7} =$ ______, $(a \neq 0)$ — $\frac{2}{a^4}$

31 $\frac{2^{16}}{2^{12}} = 2^{\square} =$ ______ — 2^4, 16

32 $\frac{2^{12}}{2^{16}} = \frac{1}{2^{\square}} =$ ______ — $\frac{1}{2^4}$, $\frac{1}{16}$

33 $\frac{2^4}{2^4} =$ ______ — 1

34 $\frac{-4x^5}{12x^2} =$ ______, $(x \neq 0)$ — $-\frac{x^3}{3}$

Simplify each of the following, applying the properties of exponents which you have learned. Assume that no variable has the value 0.

35 $\frac{2x^6}{2^3x^2} = \frac{x^{6-2}}{2^{3-1}} =$ ______ — $\frac{x^4}{2^2}$, or $\frac{x^4}{4}$

36 $\frac{3^2b^6}{3b^4} = 3^{2-1}b^{6-4} =$ ______ — $3b^2$

37 $\frac{5b^4}{5b^4} =$ ______ — 1

38 $\frac{4^2a}{4a^2} =$ ______ — $\frac{4}{a}$

39 $\frac{(5x)(5x)}{5^3x^3} = \frac{5^2x^2}{\square} =$ ______ — $\frac{5^2x^2}{5^3x^3}$, $\frac{1}{5x}$

40 $\frac{(5x)(5x)}{5x} =$ ______ — $5x$

Can the fraction $\frac{x^2}{y^3}$, $(y \neq 0)$ [illegible]

41 Note that $\frac{x^2}{y^3}$ means $\frac{x \cdot x}{\square}$.

42 The powers x^2 and y^3 have [illegible]

So you probably guessed that $\frac{x^2}{y^3}$ [illegible]

In simplifying $\frac{xy^3}{x^2y^2}$, $x \neq 0$, $y \neq$ [illegible]

[illegible]

43 Now $\frac{x}{x^2} =$ ______

44 $\frac{y^3}{y^2} =$ ______

45 Therefore, $\frac{xy^3}{x^2y^2} = \frac{1}{x} \cdot y =$ ______.

Simplify each fraction. Assume [illegible] the value 0.

46 $\frac{6ax^7}{-2x^3} =$ ______

47 $\frac{3a^2m}{4am^2} =$ ______

48 $\frac{42x^3y}{14xy^3} = \frac{(7)(2)(\ )x^3y}{(\ )(2)xy^3} =$ ______

In the last item we might have [illegible] the greatest common factor [illegible] have written:

49 $\frac{42x^3y}{14xy^3} = \frac{(\ )(3)x^3y}{14xy^3} =$ ______

When possible take advantage of greatest common factors in multiplying the following: (assume no variable 0)

50 $\frac{36a^2b^3}{8a^5b} = \frac{(\ )(9)a^2b^3}{(4)(2)a^5b} = $ ______ $\frac{(4)(9)a^2b^3}{(4)(2)a^5b}$, $\frac{9b^2}{2a^3}$

51 $\frac{81ax^2}{16a^4x^9} = $ ______ $\frac{81}{16a^3x^7}$

52 $\frac{22a^5b^5c^5}{16a^5b^2cy^2} = $ ______ $\frac{11b^3c^4}{8y^2}$

53 $\frac{x^{2a}}{x^a} = x^{\square - \square} = $ ______ $x^{2a-a} = x^a$

54 $(x^{2a})(x^a) = $ ______ x^{3a}

55 Decide whether each sentence is true or false. Then choose the response that lists all the true sentences.

M. $\frac{3^2}{2^2} = \frac{3}{2}$ P. $(\frac{4^3}{3^3})(\frac{3}{4})^3 = 1$

N. $\frac{6^3}{3^3} = 2$ Q. $\frac{6^3}{3^3} = 2^3$

O. $\frac{3^4}{2^4} = (\frac{3}{2})^4$

[A] M, O, and P [C] N, O, and Q

[B] M, P, and Q [D] O, P, and Q

Since M and N are the only false sentences, [D] is the correct choice.

If you did not see why [D] was the correct choice for Item 55, compare your work with the following:

M. $\frac{3^2}{2^2} = \frac{9}{4}$ and $\frac{9}{4} \neq \frac{3}{2}$. P. $(\frac{4^3}{3^3})(\frac{3}{4})^3 = (\frac{4}{3})^3(\frac{3}{4})^3 = \frac{4^3 \cdot 3^3}{3^3 \cdot 4^3} = 1$

N. $\frac{6^3}{3^3} = \frac{(3\cdot 2)^3}{3^3} = \frac{3^3 \cdot 2^3}{3^3} = 2^3$ and $2^3 \neq 2$. Q. $\frac{6^3}{3^3} = \frac{(2\cdot 3)^3}{3^3} = \frac{2^3\cdot 3^3}{3^3 \cdot 4^3} = 1$

O. $\frac{3^4}{2^4} = \frac{3\cdot 3\cdot 3\cdot 3}{2\cdot 2\cdot 2\cdot 2} = \frac{3}{2}\cdot\frac{3}{2}\cdot\frac{3}{2}\cdot\frac{3}{2} = (\frac{3}{2})^4$

We have been very careful to say: Assume that no variable has the value zero. Do you know the reason?

56 In $\frac{a}{b}$, which variable cannot be zero? ______ (a,b)

Answer: b

57 If b is 0, then $\frac{a}{b}$ ______ (is, is not) a number.

Answer: is not

In the rest of this chapter we shall expect you to assume that no variable has the value zero, and shall not continue to say it each time.

14-3. Non-Positive Integers as Exponents

So far we have defined powers of the form a^n, where n is a positive integer. To simplify the fraction

$$\frac{a^m}{a^n}, \quad a \neq 0$$

we needed to consider three cases.

We have seen:

1 if $m > n$, $\frac{a^m}{a^n} =$ ______

Answer: $\frac{a^m}{a^n} = a^{m-n}$

2 if $m = n$, $\frac{a^m}{a^n} =$ ______

Answer: $\frac{a^m}{a^n} = 1$

3 if $m < n$, $\frac{a^m}{a^n} =$ ______

Answer: $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$

Can we extend the notion of exponents so that the three statements listed above can be replaced by a single statement?

We know that for every non-zero a

4 $\frac{a^7}{a^4} = a^{7-4} =$ ______

Answer: a^3

5 $\frac{a^7}{a^5} = a^{7-5} =$ ______

Answer: a^2

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6	$\frac{a^{[illegible]}}{a^{[illegible]}} = a^{[illegible]} = \underline{a^{\square}} = a.$	a^1
	[illegible], these examples all illustrate,	
7	If $m > n$, then $\frac{a^m}{a^n} = \underline{a^{\square}}.$	a^{m-n}

	What would happen if we applied $$\frac{a^m}{a^n} = a^{m-n} \quad (a \neq 0)$$ in cases where m is <u>not</u> greater than n?	
	If $m = n$, then applying $\frac{a^m}{a^n} = a^{m-n}$ we have:	
8	$$\frac{a^7}{a^7} = a^{7-7} = \underline{a^{\square}}.$$	a^0
9	We know, however, that $\frac{a^7}{a^7} = $ ______.	1
	Since $\frac{a^7}{a^7} = 1$ is certainly true, and since $\frac{a^7}{a^7} = a^0$, if we use $\frac{a^m}{a^n} = a^{m-n}$, does it not seem reasonable to <u>define</u> a^0 as 1 ?	
10	Then, for example, $\frac{a^4}{a^4} = \underline{a^{\square - \square}} = \underline{a^{\square}}.$	a^{4-4}, a^0
	This is consistent with the fact that $\frac{a^4}{a^4} = 1$ if $a \neq 0$.	

Let us adopt the <u>definition</u>:

$$\text{If } a \neq 0, \text{ then } a^0 = 1.$$

	What would happen if we used $$\frac{a^m}{a^n} = a^{m-n} \quad (a \neq 0)$$ when $m < n$?	
11	We would have, for example, $\frac{a^7}{a^9} = a^{7-9} = $ ______.	a^{-2}
12	We know, however, that $\frac{a^7}{a^9} = $ ______.	$\frac{1}{a^2}$

Since $\frac{a^7}{a^9} = \frac{1}{a^2}$ is certainly true, and since $\frac{a^7}{a^9} = a^{-2}$, if we use $\frac{a^m}{a^n} = a^{m-n}$, again, does it not seem reasonable to define a^{-2} as $\frac{1}{a^2}$?

13 $\frac{a^3}{a^7} = \underline{a^{\boxed{\;-\;}}} = \underline{a^{\square}}$ — a^{3-7}, a^{-4}

14 $\frac{a^3}{a^7} = \underline{\frac{1}{\square}}$. — $\frac{1}{a^4}$

Let us adopt the <u>definition</u>:

If $a \neq 0$, and if n is a positive integer,

$$a^{-n} = \frac{1}{a^n} .$$

Now the symbols a^0, a^{-2}, a^{-4}, are meaningful. Moreover, in our examples the use of the property

$$\frac{a^m}{a^n} = a^{m-n}$$

without the restriction "$m > n$" gives results which are consistent with our earlier knowledge.

15 Thus, $\frac{a^6}{a^6} = \underline{a^{\boxed{6-\;}}}$ — a^{6-6}

16 $= \underline{a^{\square}}$ — a^0

17 $=$ ______ — 1

18 Also, $\frac{a^2}{a^5} = \underline{a^{\boxed{2-\;}}}$ — a^{2-5}

19 $= \underline{a^{\square}}$ — a^{-3}

20 $=$ ______ — $\frac{1}{a^3}$

According to the definitions of a^0 and a^{-n},

21 $16^0 =$ ______ $16^0 = 1$

22 $7^{-4} =$ ______ $7^{-4} = \frac{1}{7^4}$

23 $\frac{x^5}{(3x)^0} = \frac{x^5}{\square} =$ ______ $\frac{x^5}{1}$, x^5

24 $x^{-5} =$ ______ $\frac{1}{x^5}$

For non-zero a, we have defined:

$$a^0 = 1; \text{ and,}$$

if n is a positive integer,

$$a^{-n} = \frac{1}{a^n}.$$

Our definitions were suggested by our wish to have it true that, for <u>all</u> positive integers m and n,

$$\frac{a^m}{a^n} = a^{m-n} \text{ where } a \neq 0.$$

Based on the definitions for a^0 and a^{-n} we now prove:

if $a \neq 0$ and if m and n ar. <u>any</u> positive integers, then

25 $$\frac{a^m}{a^n} = \underline{\qquad}$$ $\frac{a^m}{a^n} = a^{m-n}$

First, suppose $m > n$.

Then we have already shown, in Section 14-2, that

$$\frac{a^m}{a^n} = a^{m-n}.$$

Second, suppose $m = n$.

Then since for any real number x, except 0, $\frac{x}{x} = 1$, we have

26 $$\frac{a^m}{a^n} = \frac{a^m}{a^m} = \underline{\qquad}.$$ 1

27 But $a^{m-m} = a^{\square}$ a^0

28 $= \underline{\qquad}$, from the definition of a^0. 1

Hence, when $m = n$

$$\frac{a^m}{a^n} = a^{m-n} \quad \text{since} \quad \frac{a^m}{a^m} = a^{m-m} = 1.$$

Finally, suppose $m < n$.

We learned in Section 14-2 that

29 $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$, where $n - m$ is ________ (positive, negative). | positive

Thus, if $m < n$, we have:

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$$

$= a^{-(n-m)}$, by our definition of negative exponents, since $n-m$ is positive.

30 $= a^{\square}$ | a^{m-n}

Hence, we have proved: for $m < n$

$$\frac{a^m}{a^n} = a^{m-n}.$$

Thus, we see that the generalization

$$\frac{a^m}{a^n} = a^{m-n}$$

holds if m and n are any positive integers and $a \neq 0$. As a matter of fact, we shall see shortly that it holds for all integers.

Simplify each of the following, using the property $\frac{a^m}{a^n} = a^{m-n}$. Express the result in terms of positive exponents only.

31 $\frac{3^5}{3^8} = 3^{\square} = \frac{1}{\square}$ | $3^{-3} = \frac{1}{3^3}$

32 $\frac{b^4}{b^2} =$ ______ | b^2

33 $\frac{b^7}{b^{10}} =$ ______ = ______ | $b^{-3} = \frac{1}{b^3}$

34 $\frac{10^5}{10^6} =$ ______ = ______ | $10^{-1} = \frac{1}{10}$

In simplifying, you often need to remember that

35 $a^m \cdot a^n = a$ ______. — a^{m+n}

Write each of the following in simplest form using non-negative exponents.

36 $\frac{10^5 \cdot 10^2}{10^8} = \frac{10^{\square}}{10^8} = 10^{\square} =$ ______ — $\frac{10^7}{10^8} = 10^{-1} = \frac{1}{10}$

37 $\frac{10^4 \cdot 10^3}{10^2 \cdot 10^5} = 10^{\square} =$ ______ — $10^0 = 1$

38 $\frac{3m^4}{m^9} =$ ______ — $\frac{3}{m^5}$

39 $\frac{t^3}{3t^5} =$ ______ — $\frac{1}{3t^2}$

40 $\frac{a^4b^3}{a^7b} = \frac{1}{a^{\square}} \cdot b^{3-1}$ — $\frac{1}{a^{7-4}} \cdot b^{3-1}$

$= \frac{1}{a^3} \cdot b^2$

41 $=$ ______ — $\frac{b^2}{a^3}$

You should see that it is easy to go directly to the final step.

$\frac{36x^2y^4}{8x^5y} = \frac{4 \cdot 9 \cdot y^{4-1}}{4 \cdot 2 \cdot x^{5-2}}$

42 $=$ ______ — $\frac{9y^3}{2x^3}$

43 $\frac{25vw^3z}{5v^4w^6} =$ ______ — $\frac{5z}{v^3w^3}$

44 $\frac{24x^3y^2c}{16xy^4c} =$ ______ — $\frac{3x^2}{2y^2}$

45 $\frac{-35mn}{7m^5} =$ ______ — $-\frac{5n}{m^4}$

46 $\frac{39a^3b^3}{-39a^3b^3} =$ ______ — -1

We have said: If n is a positive integer and if $a \neq 0$, then

$$a^{-n} = \frac{1}{a^n}.$$

You might wonder: What if n is an integer which is not positive, is it still true that $a^{-n} = \frac{1}{a^n}$? Let us see.

Suppose n is 0.

Since $-0 = 0$, then $a^{-0} = a^0$.

47 $a^0 = 1$, so $a^{-0} =$ ____. 1

48 $\frac{1}{a^0} =$ ____ 1

Hence, $a^{-0} = \frac{1}{a^0}$.

Suppose n is negative. Let us take -3 as an example.

If $a^{-n} = \frac{1}{a^n}$ is true for negative values of n, then

49 it must be true that $a^{-(-3)} = \frac{1}{a^{-3}}$; that is, $a^{\square} = \frac{1}{a^{-3}}$. a^3

Is it true that $a^3 = \frac{1}{a^{-3}}$? Yes. Let us see why.

$\frac{1}{a^{-3}} = \frac{1}{\frac{1}{a^3}}$ since $a^{-3} = \frac{1}{a^3}$

50 $a^{\square}$ the reciprocal of the reciprocal of x is x where $x \neq 0$. a^3

Thus, $a^3 = \frac{1}{a^{-3}}$ and since $a^3 = a^{-(-3)}$ we may write

51 this as $a^{-(-3)} = \frac{1}{a^{\square}}$, which is what we expected. $a^{-(-3)} = \frac{1}{a^{-3}}$

Items 49-51 suggest the generalization: If n is any integer and $a \neq 0$

$$a^{-n} = \frac{1}{a^n}.$$

Since a^{-n} and a^n are reciprocals, it also follows that, for any integer

$$a^n = \frac{1}{a^{-n}}, \ (a \neq 0).$$

If you wish, try to prove that: If $a \neq 0$, $a^{-n} = \frac{1}{a^{n}}$ is true for negative integers. Then turn to page v for help or comparison.

52	$\frac{1}{a^{-4}} = a^{-(\square)} =$ ______	$a^{-(-4)}$, a^{4}
53	$\frac{1}{x^{-2}} =$ ______	x^{2}
54	[illegible] ______	[illegible]
55	[illegible] $= \frac{1}{\square}$	$\frac{1}{a^{-[illegible]}}$
56	$y^{3} = \frac{1}{\square}$	$\frac{1}{y^{-3}}$
57	$\frac{1}{x^{4}} =$ ______	x^{-4}

Here is another thing you may have wondered about. We proved (Section 14-2) that if m and n are positive integers (and $a \neq 0$), then

$$a^{m} \cdot a^{n} = a^{m+n}.$$

But suppose m, n, or both are not positive. Can we still say that

$$a^{m} \cdot a^{n} = a^{m+n} ?$$

In other words, is the statement true for any integers m and n ?

It is true. Let us look at some examples.

	Consider, for example, $a^{-3} \cdot a^{5}$.	
58	$a^{-3} \cdot a^{5} = \frac{a^{5}}{a^{\square}}$	$\frac{a^{5}}{a^{3}}$
59	$=$ ______.	a^{2}
60	Also, $a^{-3+5} = a^{\square}$.	a^{2}
	Hence, $a^{-3} \cdot a^{5} = a^{-3+5}$.	
	Similarly, $a^{-3} \cdot a^{-6} = \frac{1}{a^{3} \cdot a^{6}}$	
61	$= \frac{1}{\square}$	$\frac{1}{a^{9}}$
62	$= a^{\square}$	a^{-9}

63 And, moreover, $a^{-3-6} = \underline{a^{\Box}}$. a^{-9}

Thus, $a^{-3} \cdot a^{-6} = a^{-3-6}$.

By now it should seem likely to you that if m and n

64 are <u>any</u> integers, $\underline{a^m \cdot a^n = a^{\Box}}$. $a^m \cdot a^n = a^{m+n}$

In fact, we can prove: if $a \neq 0$

$$a^m \cdot a^n = a^{m+n}$$

for <u>all</u> integers m and n.

If you would like to prove this statement complete Item *65.

* We already know that if m and n are positive integers, then

$$a^m \cdot a^n = a^{m+n}.$$

To show that the statement holds for <u>all</u> integers m and n, we would need to consider several cases:

One of the numbers m and n positive and the other negative.

Both m and n negative.

One or both zero.

Let's do the case where m is positive and n is negative.

*65 Then $a^m \cdot a^n = a^m \cdot \frac{1}{a^{-n}}$ ($-n$ is ______) positive

$$= \frac{a^m}{a^{-n}}$$

$$= a^{m-(-n)} \quad \text{(since } -n \text{ is positive)}$$

*66 $= \underline{a^{\Box}}$ a^{m+n}

The other cases are just as easy.

One final question you might ask: Is it also true that

$$\frac{a^m}{a^n} = a^{m-n}$$

for <u>all</u> integers m and n ? By now you have had enough experience in this section with this kind of question that you can guess the answer. It is "yes".

For example, we see that

$\frac{x^4}{x^{-6}} = x^4 \cdot \frac{1}{x^{-6}}$ Meaning of division

67 $= x^4 \cdot \underline{x^{\square}}$ Since $\frac{1}{x^{-n}} = x^n$ x^6

68 [illegible] $= \underline{x^{\square}}$ $x^m \cdot x^n = x^{m+n}$ x^{10}

If we [illegible] $\frac{a^m}{a^n} = a^{m-n}$ for all integers m and n

we see that

69 $\frac{x^4}{x^{-6}} = x^{4-(-6)}$ which also equals $\underline{x^{\square}}$. x^{10}

Were we to take the time to prove

$$\frac{a^m}{a^n} = a^{m-n} \text{ for all integers, } a \neq 0,$$

this example would give us an idea for the proof.

In simplifying expressions involving exponents we can use the following generalizations for all integers m and n: If $a \neq 0$,

$$a^m \cdot a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$a^{-n} = \frac{1}{a^n}$$

$$a^0 = 1$$

Often you will see more than one way of proceeding, but your result will be the same whatever your choice.

70 Recall, too, that $(\frac{a}{b})(\frac{c}{d}) = \frac{\square}{\square}$. $\frac{ac}{bd}$

71 In particular, $\frac{1}{x} \cdot \frac{1}{y} = \frac{1}{\square}$. $\frac{1}{xy}$

In many problems, alternate approaches to simplifying expressions are possible, and sometimes one approach may lead to less work than another. Recognizing the alternatives comes with practice and experience, so at times we shall try to show more than a single approach.

Simplify and write with positive exponents, assuming no variable to have a value of 0. (Note that more than one approach is shown for simplifying each of the following three expressions.)

72 $x^{-5} \cdot x^{2} = \underline{x^{\square}} = \underline{\qquad}$ — x^{-3}, $\frac{1}{x^{3}}$

73 $x^{-5} \cdot x^{2} = \underline{\frac{1}{x^{\square}} \cdot x^{2}} = \frac{x^{2}}{x^{5}} = \underline{\qquad}$ — $\frac{1}{x^{5}} \cdot x^{2}$, $\frac{1}{x^{3}}$

74 $\frac{m^{-3}}{m^{-5}} = \underline{m^{-3-\square}} = \underline{\qquad}$ — $m^{-3-(-5)}$, m^{2}

75 $\frac{m^{-3}}{m^{-5}} = m^{-3} \cdot \frac{1}{m^{-5}} = \frac{1}{m^{3}} \cdot m^{5} = \underline{\frac{m^{\square}}{m^{\square}}} = m^{2}$. — $\frac{m^{5}}{m^{3}}$

In the last items you may have seen that it was not necessary to write down all the steps shown. In fact, we could have written

76 $\frac{m^{-3}}{m^{-5}} = \frac{m^{5}}{m^{3}} = \underline{\qquad}$ — m^{2}

77 $\frac{a^{-4}}{a^{-8}} = \underline{\frac{a^{8}}{a^{\square}}} = \underline{\qquad}$ — $\frac{a^{8}}{a^{4}}$, a^{4}

78 $\frac{a^{-4}}{a^{-8}} = \underline{a^{-4-\square}} = \underline{\qquad}$ — $a^{-4-(-8)}$, a^{4}

Let us simplify $\frac{x^{-2}y^{-3}}{x^{4}y^{-2}}$ $(x \neq 0,\ y \neq 0.)$

A possible way of doing this is:

$$\frac{x^{-2}y^{-3}}{x^{4}y^{-2}} = \frac{x^{-2}}{x^{4}} \cdot \frac{y^{-3}}{y^{-2}}$$

79 $$= \underline{\frac{1}{x^{4-(\square)}} \cdot \frac{1}{y^{-2-(\square)}}}$$ — $\frac{1}{x^{4-(-2)}} \cdot \frac{1}{y^{-2-(-3)}}$

80 $$= \underline{\frac{1}{x^{\square}} \cdot \frac{1}{y^{\square}}}$$ — $\frac{1}{x^{6}} \cdot \frac{1}{y^{1}}$

$$= \frac{1}{x^{6}y}$$

Find the simplest name for

81 $\frac{10^3 \cdot 10^{-4}}{10^{-5}} = \frac{10^{\square}}{10^{\square}}$ — $\frac{10^{-1}}{10^{-5}}$

82 $= 10^{\square}$ — $10^{-1-(-5)}$

83 $=$ ______ — 10^4

Simplify each of the following to a form with only positive exponents. If you have difficulty with any of them, look at the corresponding item among Items 92 to 99, where you will find help.

84 $\frac{10^4 \times 10^{-2}}{10^2}$ — 1

85 $\frac{10^3 \times 10^2}{10^{-2}}$ — 10^7

86 $.007 \times 10^4 \times 10^{-4}$ — $.007$

87 $\frac{12a^4b}{3a^7b^2}$ — $\frac{4}{a^3b}$

88 $\frac{2x^2y^{-2}}{4^2x^2y^2}$ — $\frac{1}{8y^4}$

89 $\frac{3^2 \times 2^{-3}}{2^3 \times 3^{-2}}$ — $\frac{3^4}{2^6}$, or $\frac{81}{64}$

90 $\frac{10^3 \times 10^{-4} \times 10^0}{10^2 \times 10^{-3}}$ — 1

91 $\frac{2^{-3}x^{-2}y^4}{2^{-2}x^2y^{-1}}$ — $\frac{y^5}{2x^4}$

For Item 84.

92 $\frac{10^4 \times 10^{-2}}{10^2} = \frac{10^4}{10^2 \cdot 10^2} = \frac{10^4}{\square} = 1$ — $\frac{10^4}{10^4}$

For Item 85.

93 $\frac{10^3 \times 10^2}{10^{-2}} = 10^3 \cdot 10^2 \cdot 10^2 =$ ______ — 10^7

For Item 86.

94 $0.007 \times 10^4 \times 10^{-4} = \underline{0.007(10^{\square})}$ $= 0.007(1) = 0.007$ $0.007(10^0)$

For Item 87.

95 $\frac{12a^4b}{3a^7b^2} = \underline{\frac{12}{3a^{7-4}b^{\square}}} = \frac{4}{a^3b}$ $\frac{12}{3a^{7-4}b^{2-1}}$

For Item 88.

96 $\frac{2x^2y^{-2}}{4^2x^2y^2} = \underline{\frac{2x^2}{2^{\square}x^2y^2y^{\square}}} = \frac{1}{8y^4}$ $\frac{2x^2}{2^4x^2y^2y^2}$

For Item 89.

97 $\frac{3^2 \cdot 2^{-3}}{2^3 \cdot 3^{-2}} = \underline{\frac{3^2 \cdot 3^{\square}}{2^{\square} \cdot 2^{\square}}} = \frac{3^4}{2^6}$ $\frac{3^2 \cdot 3^2}{2^3 \cdot 2^3}$

For Item 90.

98 $\frac{10^3 \times 10^{-4} \times 10^0}{10^2 \times 10^{-3}} = \frac{10^3 \times 10^3 \times 10^0}{10^2 \times 10^4} = \underline{\frac{10^{\square}}{10^{\square}}} = 1$ $\frac{10^6}{10^6}$

For Item 91.

$\frac{2^{-3}x^{-2}y^4}{2^{-2}x^2y^{-1}} = \frac{1}{2^{(-2+3)}} \cdot \frac{1}{x^{(2+2)}} \cdot \frac{y^{(4+1)}}{1}$

99 $= \underline{\frac{y^{\square}}{2^{\square}x^{\square}}} = \frac{y^5}{2x^4}$ $\frac{y^5}{2^1x^4}$

14-4. Using Exponents

1	$x^3 = x \cdot$ ____	$x \cdot x \cdot x$
	Similarly,	
	$(ab)^3 = (ab)(ab)(ab)$	
2	$= (aaa)($ ____ $)$	$(aaa)(bbb)$
3	$=$ ____	a^3b^3
4	$(xy)^5 = x^{\square}y^{\square}$	x^5y^5
5	$(3x)^3 = 3^{\square}x^{\square}$	3^3x^3
6	$= 27x^{\square}$	$27x^3$
7	In general, $(ab)^n =$ ____.	a^nb^n
8	$(\frac{a}{b})^2 = (\ \ )(\ \ ) = \frac{a^{\square}}{b^{\square}}$	$(\frac{a}{b})(\frac{a}{b})$, $\frac{a^2}{b^2}$
9	$(\frac{x}{y})^3 = (\ \ )(\ \ )(\ \ ) = \frac{x^{\square}}{y^{\square}}$	$(\frac{x}{y})(\frac{x}{y})(\frac{x}{y})$, $\frac{x^3}{y^3}$
10	$(\frac{m}{n})^5 =$ ____	$\frac{m^5}{n^5}$
11	In general, $(\frac{a}{b})^n =$ ____, $b \neq 0$.	$\frac{a^n}{b^n}$
	Using the definition of exponents, we can write:	
12	$(a^2)^4 = (\ \ )(\ \ )(\ \ )(\ \ )$	$(a^2)(a^2)(a^2)(a^2)$
13	$= a^{\square}$	a^8
14	$(a^2b^3)^3 = (a^2b^3)(\ \ )(\ \ )$	$(a^2b^3)(a^2b^3)(a^2b^3)$
15	$= (a^2a^2a^2)($ ____ $)$	$b^3b^3b^3$
16	$= a^{\square}b^{\square}$	a^6b^9
	By this time you should begin to see a general pattern emerging.	

Let us look at one more example.

17 In forming m^3, we use m as a factor $\underset{\text{(how many)}}{\underline{\qquad}}$ times. — three

18 In $(m^3)^4$ we use m^3 as a factor _____ times. — four

We conclude: In $(m^3)^4$ we must use m as a factor

19 $4 \times$ _____, or 12, times. — 4×3

$$(m^3)^4 = \underbrace{(\underbrace{m \cdot m \cdot m}_{3})(\underbrace{m \cdot m \cdot m}_{3})(\underbrace{m \cdot m \cdot m}_{3})(\underbrace{m \cdot m \cdot m}_{3})}_{4 \times 3}$$

Hence, $(m^3)^4 = m^{12}$

20 Likewise, we can see that $(y^5)^7 = y^{\square}$. — y^{35}

Our examples demonstrate that, in general,

$$(a^m)^n = a^{mn}.$$

The properties of exponents developed in the preceding items are

21 $(a^m)^n = a^{\square}$ — $(a^m)^n = a^{mn}$

22 $(\frac{a}{b})^n = \frac{\square}{\square}$, $(b \neq 0)$ — $(\frac{a}{b})^n = \frac{a^n}{b^n}$

23 $(ab)^n = a^{\square}b^{\square}$. — $(ab)^n = a^n b^n$

Use these properties in the following items:

24 $(a^3)^5$ = _____ — a^{15}

25 $(x^a)^b$ = _____ — x^{ab}

26 $(y^3)^2$ = _____ — y^6

27 $(\frac{x}{y})^4$ = _____, $(y \neq 0)$ — $\frac{x^4}{y^4}$

28 $(\frac{a^2}{2x})^3$ = _____, $(x \neq 0)$ — $\frac{a^6}{2^3x^3}$, or $\frac{a^6}{8x^3}$

29 $\left(\frac{a^3}{y^4}\right)^5$ = _____, $(y \neq 0)$ — $\frac{a^{15}}{y^{20}}$

The [illegible]

$$(a^{m})^{n} = a^{mn}$$

$$\left(\frac{a}{b}\right)^{n} = \frac{a^{n}}{b^{n}}, \quad b \neq 0$$

$$(a^{m})^{n} = a^{m \cdot n}$$

[illegible] where [illegible] is a positive integer. You would [illegible] for any integers m and n? [illegible] "Yes." So it is. You may wish [illegible].

30	$\left(\frac{[illegible]}{[illegible]}\right)^{[illegible]}$ = ______, [illegible] $\neq$ [illegible].	$\frac{7^5a^{10}}{3^5b^{15}}$
31	$(x^{[illegible]}y^{[illegible]})^{[illegible]}$ = ______,	x^8y^{12}
32	$3(a^{3}b^{[illegible]})^{[illegible]}$ = ______,	$3a^9b^{15}$
33	$(3a^{[illegible]}b^{[illegible]})^{3}$ = ______.	$3^3a^9b^{15}$

34 Consider the following fractions:

R. $\frac{(-3)^2a}{9}$ S. $\frac{-3^2a}{9}$ T. $\frac{(-3a)^2}{9}$ U. $\frac{(-3a)^3}{9}$

If $a > 0$, which fractions are positive numbers?

[A] R and S
[B] R and T
[C] all except S

R. $\frac{(-3)^2a}{9} = \frac{9a}{9} = a$

S. $\frac{-3^2a}{9} = \frac{-(3)^2a}{9} = \frac{-9a}{9} = -a$

T. $\frac{(-3a)^2}{9} = \frac{(-3a)(-3a)}{9} = \frac{9a^2}{9} = a^2$

U. $\frac{(-3a)^3}{9} = \frac{-27a^3}{9} = -3a^3$

So the correct response is [B].

35 Which of the following are true if x is any non-zero real number and a is any integer?

P. $\dfrac{x^{2a}}{x^{a}} = x^{a}$ Q. $x^{2a} \cdot x^{a} = x^{3a}$ R. $(x^{2a})^3 = x^{5a}$

[A] P and Q

[B] P and R

[C] all three are true

$$\frac{x^{2a}}{x^{a}} = x^{2a-a} = x^{a}$$

$$x^{2a} \cdot x^{a} = x^{2a+a} = x^{3a}$$

$$(x^{2a})^3 = x^{2a} \cdot x^{2a} \cdot x^{2a} = x^{6a} \quad \text{or} \quad (x^{2a})^3 = x^{3 \cdot 2a} = x^{6a}$$

Since $x^{6a} \neq x^{5a}$, the correct choice is [A].

Simplify the following. Your responses should contain positive exponents only. If you have difficulty with any item, or if you get an incorrect answer, look at the corresponding item among Items 48 to 59, where you will find hints. (Assume that no variable has the value 0.)

36	$\dfrac{5x^2}{15xy^2} =$ ______	$\dfrac{x}{3y^2}$
37	$\dfrac{(5x)^2}{15xy^2} =$ ______	$\dfrac{5x}{3y^2}$
38	$\dfrac{5x^2}{15(xy)^2} =$ ______	$\dfrac{1}{3y^2}$
39	$\dfrac{(2y^2)^5}{(2y^2)^5} =$ ______	1
40	$\dfrac{(2y)^5}{2y^5} =$ ______	2^4, or 16
41	$\dfrac{-7^2z^{15}}{49z^{30}} =$ ______	$-\dfrac{1}{z^{15}}$

[illegible]

[illegible]

y^2

[illegible]

[illegible]

[illegible]

[illegible]

[illegible] $\frac{x}{3y^2}$

[illegible] $\frac{5x}{3y^2}$

50 [illegible] $\frac{1}{3y^2}$

[illegible] ([illegible], for any real number $a \neq 0$.)

For Item 4[illegible].

51 [illegible] 2^4, or 16

For Item 41.

$$\frac{-7^2 z^{15}}{49 z^{30}} = \frac{-(7^2) z^{15}}{7^2 z^{30}}$$

$$= -\frac{1}{z^{30-15}}$$

52 $= \_\_\_\_\_\_$ $\quad -\frac{1}{z^{15}}$

For Item 42.

$$\frac{(-7)^2 z^{15}}{49 z^{30}} = \frac{49 z^{15}}{49 z^{30}}$$

53 $= \_\_\_\_\_\_$ $\quad \frac{1}{z^{15}}$

For Item 43.

54 $\frac{-7^2 z^{30}}{-49 z^{15}} = \frac{-49 z^{30}}{-49 z^{15}} = \_\_\_\_\_\_$ $\quad z^{15}$

For Item 44.

$$\frac{(2y^2)^3(2y)}{(2y)^3(2y^2)} = \frac{2^3 y^6 \cdot 2y}{2^3 y^3 \cdot 2y^2}$$

55 $= \frac{2^{\square} y^{\square}}{2^{\square} y^{\square}}$ $\quad \frac{2^4 y^7}{2^4 y^5}$

56 $= \_\_\_\_\_\_$ $\quad y^2$

For Item 45.

$$\left(\frac{32a^3}{45b^2}\right)\left(\frac{3b}{4a}\right)^2 = \frac{32a^3 \cdot 9b^2}{45b^2 \cdot 16a^2}$$

$$= \frac{16 \cdot 9 \cdot 2a^3 b^2}{16 \cdot 9 \cdot 5a^2 b^2}$$

57 $= \_\_\_\_\_\_$ $\quad \frac{2a}{5}$

For Item 46.

$$\left(\frac{63cd}{20d^2x}\right)\left(\frac{40xd^4}{7c^3}\right) = \frac{9 \cdot 7 \cdot 20 \cdot 2cd^5 x}{7 \cdot 20c^3 d^2 x}$$

58 $= \_\_\_\_\_\_$ $\quad \frac{18d^3}{c^2}$

For Item 47.

$$\left(\frac{-12abx^3}{5b^3}\right)\left(\frac{-25b^4}{4ax^4}\right) = \frac{(-1)(-1)(4)(3)(5)(5)ab^5x^3}{(4)(5)(a)(b^3)(x^4)}$$

$$= \underline{\qquad}$$

$\frac{15b^2}{x}$

In the last chapter we applied our knowledge about factoring integers to adding and subtracting fractions. We can use the ideas developed there in working with fractions in which exponents appear.

Let us see how we can write

$$\frac{5}{4x^3} - \frac{2}{3x^2} + \frac{1}{6x^4}, \ x \neq 0$$

as a single fraction.

The factorizations of the denominators are:

$$4x^3 = 2\cdot 2\cdot x\cdot x\cdot x$$

$$3x^2 = 3\cdot x\cdot x$$

60 $$6x^4 = \underline{\qquad}$$ $3\cdot 2\cdot x\cdot x\cdot x\cdot x$

61 2 appears as a factor no more than $\underset{\text{(how many)}}{\underline{\qquad\qquad}}$ time(s) in any denominator. two

3 appears as a factor no more than once in any denominator.

x, however, appears as a factor three times in $4x^3$,

62 two times in $3x^2$ and ______ times in $6x^4$. four

63 Hence, the least common denominator is $\underline{3\cdot 2^{\square}x^{\square}}$. $3\cdot 2^2x^4$

$$\frac{5}{4x^3} - \frac{2}{3x^2} + \frac{1}{6x^4} = \frac{5\cdot 3x}{4x^3\cdot 3x} - \frac{2\cdot 4x^2}{3x^2\cdot 4x^2} + \frac{1\cdot 2}{6x^4\cdot 2}$$

$$= \frac{15x - 8x^2 + 2}{12x^2}$$

Simplify $\frac{5}{3x^2} + \frac{11}{6xy} - \frac{4}{9y^2}$. Write your work neatly and carefully on your own paper. You should obtain as a result:

64 $$\frac{5}{3x^2} + \frac{11}{6xy} - \frac{4}{9y^2} = \frac{\square}{\square}, \ x \neq 0, \ y \neq 0.$$ $\frac{30y^2+33xy-8x^2}{18x^2y^2}$

If you were not correct, complete Items 65 to 71. Otherwise, omit these items, and go on to Item 72.

[illegible]

	[illegible] $\frac{\;}{\;} + \frac{\;}{\;} - \frac{\;}{\;}$, [illegible] $\neq$ [illegible]	
	[illegible]	
[illegible]	common denominators are: [illegible], [illegible], [illegible], [illegible] ______.	[illegible]
	[illegible]	
[illegible]	factor in any one denominator is ______.	[illegible]
[illegible]	y, however, appears ______ as a factor in	twice
	one denominator. It appears twice in [illegible].	
	Also, there is a denominator in which [illegible] appears as a	
[illegible]	factor ______.	twice
[illegible]	Likewise, y appears ______ in one denominator.	twice
	[illegible]	
[illegible]	$\cdot\,\square\cdot x^{\square}\cdot y^{\square}$	$\cdot x^2 \cdot y^2$
[illegible]	$\frac{\;}{3x^2} - \frac{\;}{\;xy} - \frac{\;}{y^2} = \frac{\;}{\;x^2}\cdot\frac{\square}{\;\cdot y^2} + \frac{\;}{\;xy}\cdot\frac{\;xy}{\;xy} - \frac{\;}{y}\cdot\frac{\;}{\;}$	[illegible]
[illegible]	$= \frac{\;y^2 + \square - \square}{15x^2y^2}$	$\frac{[illegible] - 54}{15x^2y^2}$

	Simplify $\frac{\;}{\;a^2} + \frac{\;}{\;ab} - \frac{7}{\;b^2}$, $a \neq 0$, $b \neq 0$.	
73	The least common denominator is ______.	[illegible] b^2
74	$\frac{\;}{\;a^2} + \frac{\;}{\;ab} - \frac{7}{\;b^2} =$ ______	$\frac{[illegible] - 245a^2}{[illegible]}$
	Simplify $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.	
75	The least common denominator is ______.	[illegible]
76	$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} =$ ______, $a \neq 0$, $b \neq 0$, $c \neq 0$.	$\frac{bc + ac + ab}{abc}$

We used the distributive property in earlier chapters in order to write indicated products as indicated sums. We shall do so now to get another opportunity to practice with exponents.

Write as an indicated sum:

$$(x^2 + 1)(x^3 + x^2 + 1)$$

77 $(x^2 + 1)(x^3 + x^2 + 1) = x^2(x^3 + x^2) + \underline{\quad}(x^3 + x^2 + 1)$ 1

78 $= \underline{x^{\square} + x^{\square} + x^{\square}} + x^3 + x^2 + 1$ $x^4 + x^2$

$= x^5 + x^4 + x^3 + 2x^2 + 1$

Write each of the following as indicated sums:

$(2a^3 - b^2)(3a^2 - 2b^2)$

79 $= 2a^3(3a^2 - 2b^2) + \underline{(\quad)(\qquad)}$ $(-b^2)(3a^2-2b^2)$

80 $= \underline{6a^5 - 4a^3b^2 \qquad}$ $6a^5 - 4a^3b^2 - 3a^2b^2 + 2b^4$

81 $(a^2 - 3)(a^2 - 2a + 1)$ $\underline{\qquad}$ $a^4 - 2a^3 - 2a^2 + 6a - 3$

82 $(2x - 3y)^2 = \underline{\qquad}$ $4x^2 - 12xy + 9y^2$

*83 $(a + b)^3 = \underline{(a + b)(a + b)^{\square}}$ $(a + b)(a + b)^2$

$= a(a + b)^2 + b(a + b)^2$

*84 $= a(a^2 + \underline{\quad} + b^2) + b(a^2 + \underline{\quad} + b^2)$ $2ab,\ 2ab$

*85 $= \underline{\qquad}$ $a^3 + 3a^2b + 3ab^2 + b^3$

86 If a is 2, b is -2, c is 3, d is -3, determine the value of each of the following. You will find it easier if you change fractions to lowest terms when possible before substituting numerical values.

[A] $-2a^2b^2c^2$

[B] $(-2abc)^2$

[C] $\dfrac{-4a^4d}{6b^2a^3}$

[D] $\dfrac{a^3 - b^3}{a^3b^3}$

[E] $\dfrac{(a + b + c)^2}{a^2 + b^2 + c^2}$

Of the values for [A], [B], [C], [D], and [E] which value is not a member of the set $\{-288, 1, 0, \frac{7}{13}, 576\}$?

The correct answer is (E). Your work might look something like this:

$-2a^2b^2c^2 = (-2)(2)^2(-2)^2(3)^2 = $ [illegible] $= -288$

$(-2abc)^2 = \left((-2)(2)(-2)(3)\right)^2 = $ [illegible]

$\frac{-4a^{-1}d}{4b^2a^2} = \frac{-2ad}{3b^2} = \frac{(-2)(2)(-3)}{[illegible]\,3(-2)^2} = $ [illegible]

$\frac{a^2 \cdot [illegible]^3}{a^2b^2} = \frac{2^3 + (-2)^3}{2^3(-[illegible])} = 0$

$\frac{(a+b+c)^2}{a^2+b^2+c^2} = \frac{(2-2+[illegible])^2}{2^2+(-2)^2+[illegible]^2} = $ [illegible]

14-5. Summary and Review

Summary

Exponents were first introduced with positive integers. If n is a positive integer, then a^n [illegible] n times.

$$\underbrace{a^n = a \cdot a \;\ldots\; \cdot a}_{n \text{ factors}}$$

By definition, $a^{-n} = \frac{1}{a^n}$ (n positive and $a \neq 0$)

By definition, $a^0 = 1$ $(a \neq 0)$

We established the following generalizations which hold for all integers m and n:

$$a^m \cdot a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}, \quad (a \neq 0)$$

$$a^{-n} = \frac{1}{a^n} \text{ and } \frac{1}{a^{-n}} = a^n, \quad (a \neq 0)$$

$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, \quad (b \neq 0)$$

$$(a^m)^n = a^{mn}$$

In all of these statements we must note the restriction that division by 0 is not allowed.

The answers to the following review problems are on page [illegible].

Review Problems

1. Find the following sums:

   (a) [illegible] (c) $\frac{[illegible]}{5a^2} + \frac{[illegible]}{12a} + \frac{[illegible]}{[illegible]}$

   (b) [illegible] (d) [illegible]

2. Simplify:

   (a) [illegible] (b) [illegible]

   (c) [illegible]

3. What is the first prime number after 123 ?

4. Four times an integer is ten more than twice its successor. What is the integer?

5. If the domain of the variable is the set of prime numbers, find the truth set of the following:

   (a) [illegible] (c) $3x^2 < 123$

   (b) [illegible] $= 12 +$ [illegible]x (d) $|x - 10| < 3$

6. Simplify:

   (a) [illegible] (d) $\frac{2^2 \cdot 2^5}{2^4}$

   (b) $\frac{[illegible]xy}{3x^2}$ (e) $\frac{3^5 a^4}{3^2 a}$

   (c) $m^{[illegible]} \cdot m^{-2}$ (f) $\frac{10^3 \times 10^{-1}}{10^{-2}}$

7. Write the indicated products as indicated sums.

(a) $a^2(a + 1)$

(b) $xy^2(x^2 + y^3)$

(c) $(2x + 1)3x^2$

(d) $(-mn)(m - n)$

(e) $(a^2 + b^2)(a + b)$

(f) $x^{-1}(x^2 + x^3)$

(g) $(a + b)(a^{-1} + b^{-1})$

8. If n is a positive integer, which of the following are even numbers, which are odd, and which may be either?

(a) n^2

(b) n^3

(c) $2n$

(d) $2n + 1$

(e) $(2n)^2$

(f) $(2n + 1)^2$

(g) $4n^2$

(h) $2n - 1$

(i) $2^{10} + 3^{10}$

(j) $2^{10} + 6^{10}$

9. Which of the following are non-negative for any real number n ?

(a) n^2

(b) $(-n)^3$

(c) $(-n)(-n)$

(d) $-n^3$

(e) $(-n)^2$

(f) $(n^2)(n^2)$

(g) n^4

(h) $(-n)^4$

(i) $-n^4$

(j) $-|n^2|$

(k) $|-n^3|$

10. Two squares differ in area by 27 square units. A side of the larger is one unit longer than a side of the smaller. Write and solve an equation to find the length of the side of the smaller square.

11. For 27 days Bill has been saving nickels and dimes for summer camp expenses. He finds he has 41 coins, the value of which is $3.35. If he has more dimes than nickels, how many nickels does he have?

12. A 100 gallon container is tested and found to contain 15% salt. How much of the 100 gallons should be withdrawn and replaced by pure water to make a 10% solution?

13. A jet travels 10 times as fast as a passenger train. In one hour the jet will travel 120 miles further than the passenger train will go in [illegible] hours. What is the rate of the jet? the train?

14. Two trains 320 miles apart travel towards each other. One is traveling [illegible] as fast as the other. What is the rate of each if they meet in 3 hours and 12 minutes?

15. Marie's candy store made a 40 lb. mixture of creams selling at $1.00 per pound and nut centers selling at $1.40 per pound. If the mixture is to sell at $1.10 per pound, how many pounds of each kind of candy should be used?

Chapter 15

RADICALS

15-1. Roots

Let us consider the truth set of $x^2 = 49$. We recognize that $7^2 = 49$. Since it is also true that $(-7)^2 = 49$, we see that -7 is also an element of the truth set of $x^2 = 49$.

1	The truth set of $x^2 = 81$ is ______.	$\{9,-9\}$
2	If $x = 11$, then $x^2 =$ ______. A different number	121
3	whose square is 121 is ______.	-11
4	$(-15)^2 =$ ______, so -15 is a number whose square is	225
5.	225. The truth set of $x^2 = 225$ is ______.	$\{15,-15\}$
	It is true that if a is any real number, then	
	$a^2 = (-a)^2$. Thus, if b is the square of some number	
6	a, then b is also the ______ of $-a$.	square

If b is a non-negative number, then we shall denote by $\sqrt{b}$ the <u>non-negative</u> number whose square is b. Thus, $\sqrt{81} = 9$. The symbol $\sqrt{\ }$ is called the <u>radical</u> sign. Note that $-9 = -\sqrt{81}$. For any non-negative real number b, the symbol $\sqrt{b}$ thus names <u>exactly</u> <u>one</u> real number, the non-negative number whose square is b.

7	$\sqrt{64} =$ ______	8
8	$-\sqrt{144} =$ ______	-12
9	$\sqrt{\quad}$ ______ $= 10$	$\sqrt{100}$
10	______ $\sqrt{\quad}$ $= -12$	$-\sqrt{144}$
11	If $m > 0$, $\sqrt{m}$ is a ______ number.	positive
12	The truth set of $x^2 = -4$ is ______, since there	$\emptyset$
13	______ (is, is not) a real number whose square is -4.	is not
14	We conclude: $\sqrt{b}$ has meaning for us only if $b \geq$ ____.	$b \geq 0$

We can summarize: If b is a positive real number, the <u>positive</u> number whose square is b is denoted by $\sqrt{b}$. The <u>negative</u> number whose square is b is $-\sqrt{b}$. We define: $\sqrt{0} = 0$.

We often say, "The square roots of 81 are 9 and -9", meaning that the truth set of $x^2 = 81$ is $\{9,-9\}$. When we read $\sqrt{81}$ aloud we say, "the square root of 81", meaning, of course, <u>only</u> the number 9. When we use the words, "square root of 81", the context helps indicate whether you are speaking about the <u>single</u> positive number $\sqrt{81}$ or the <u>two</u> numbers, 9 and -9, which have 81 as their square. However, the symbol $\sqrt{81}$ refers <u>only</u> to the positive number 9.

	Classify each as true or false:		
15	$-\sqrt{0} = -0 = 0$	(true,false)	true
16	$(\sqrt{5})(\sqrt{0}) = 0$	______	true
17	$\sqrt{16} + \sqrt{16} = 0$	______	false
18	$\sqrt{81} - \sqrt{0} = -9$	______	false
19	$\frac{\sqrt{9}}{\sqrt{0}}$ does not name a number	______	true
20	$-\sqrt{-25} = 5$	______	false

21	$\sqrt{x^2}$ is never a ______ number.	negative
	Suppose we try some numbers for x and see what values we get for $\sqrt{x^2}$.	
22	If $x = 3$, $\sqrt{x^2} = \sqrt{3^2} = \sqrt{9} =$ ______.	3
23	If $x = -3$, $\sqrt{x^2} = \sqrt{(-3)^2} = \sqrt{9} = 3 = -(\ \ )$.	-(-3)
24	Thus, $\sqrt{x^2} =$ ______ is true when $x = 3$,	x
25	and $\sqrt{x^2} = -x$ is ______ when $x = -3$.	true
	If $x = 0$, $\sqrt{x^2} = \sqrt{0^2} = 0 = -0$.	
26	Thus, when $x = 0$, $\sqrt{x^2} =$ ______ is true,	x
27	and also $\sqrt{x^2} =$ ______ is true.	-x

We may generalize as follows:

For any real number x,

$$\sqrt{x} = x, \text{ if } x \geq 0$$
$$\sqrt{x} = -x, \text{ if } x < 0.$$

In Chapter 6 we made a similar pair of statements about $|x|$:

$$|x| = x, \quad \text{if } x \geq 0$$
$$|x| = -x, \quad \text{if } x < 0.$$

Thus, we can state:

If x is any real number,

$$\sqrt{x^2} = |x|$$

$\sqrt{x^2} = |x|$ is true if x is -3, since

28 $\sqrt{(-3)^2} = 3$ and $|-3| =$ ______. — 3

$\sqrt{x^2} = |x|$ is true if x is 3, since

29 $\sqrt{3^2} = 3$ and $|3| =$ ______. — 3

We must be sure to recall that $\sqrt{b}$ makes sense only

30 if b ______ 0. — $\geq$

31 If b is negative, we can find ______ (no, one, many) number(s) a such that $(a)(a) = b$. — no

32 Remember, for all real values of x we have $x^2 \geq$ ____. — 0

Hence, the symbol $\sqrt{x^2}$ stands for a real number for all real values of x.

33 If $a < 3$, then $a - 3$ is ______ (positive, negative). — negative

The symbol $\sqrt{a - 3}$ is the name of a real number only

34 if $a \geq$ ______. — 3

The symbol $\sqrt{x^3}$ is the name of a real number only if

35 x ______ 0. — $\geq$

Remember: If x and y are real numbers and $\sqrt{x} = y$

36 then it must be true that $x \geq$ ______ and also that — 0

37 y ______ 0. — $\geq$

For each symbol that names a real number, state a simpler name for that number.

38 $\sqrt{36} =$ ______ — 6

39 $\sqrt{-36} =$ ______ — not a real number

40 $\sqrt{(-6)^2} = |-6| =$ ______ — 6

41	$-\sqrt{36} =$ ______	-6
42	$\sqrt{64} =$ ______	8
43	$\sqrt{(-8)^2} =$ ______	8
44	$\sqrt{(4)^3} = \sqrt{64} =$ ______	8
45	$\sqrt{(-4)^3} = \sqrt{-64} =$ ______	not a real numbe
46	If $n \geq 0$, $\sqrt{n^2} =$ ______	n, or $\lvert n \rvert$
47	If $n < 0$, $\sqrt{n^2} =$ ______	$-n$, or $\lvert n \rvert$
48	If n is any real number, $\sqrt{n^2} =$ ______	$\lvert n \rvert$

49	If $n \geq 0$, $\sqrt{n^3}$ ______ (is, is not) a real number.	is
50	If $n < 0$, $\sqrt{n^3}$ ______ (is, is not) a real number.	is not

51	$\sqrt{(-9)^2} = \lvert -9 \rvert =$ ______	9
52	$\sqrt{\left(\frac{2}{3}\right)^2} = \left\lvert \frac{2}{3} \right\rvert =$ ______	$\frac{2}{3}$
53	$\sqrt{169} = \sqrt{(13)^2} = \lvert\ \ \rvert =$ ______	$\lvert 13 \rvert$, 13
54	Also, $\sqrt{169} = \sqrt{(-13)^2} = \lvert\ \ \rvert =$ ______	$\lvert -13 \rvert$, 13
55	$\sqrt{\frac{121}{25}} =$ ______	$\frac{11}{5}$

What you learned in a previous chapter about factoring may be put to use here in finding the square root of a given number.

56	$441 = 9(\ \ \ \ )$ [Note that 441 is divisible by 9, since $4 + 4 + 1 = 9$.]	9(49)
	$= (3)(3)(7)(7)$	
57	$= \big((3)(7)\big)\big((\ \ )(\ \ )\big) = \big((3)(7)\big)^2$	$\big((3)(7)\big)$
58	Hence, $\sqrt{441} = (3)(7) =$ ______.	21

59	$1936 = 4(\underline{\quad\quad})$ [Note that 4 divides 1936, since 4 divides 36.]	$4(484)$
60	$= (4)(4)(121) = (4)(4)\underline{(\quad)(\quad)}$	$(11)(11)$
61	$= \underline{((\quad)(\quad))^2}$	$((4)(11))^2$
62	Hence, $\sqrt{1936} = (4)(11) = \underline{\quad\quad}$	44
63	$\sqrt{2304} = \underline{\quad\quad}$	48
64	$\sqrt{676} = \underline{\quad\quad}$	26
65	$-\sqrt{1225} = \underline{\quad\quad}$	-35
66	$\sqrt{11.56} = \underline{\quad\quad}$	3.4
67	$\sqrt{0.0256} = \underline{\quad\quad}$	0.16

If Items 63-67 gave you trouble, do Items 68-85; otherwise, skip to Item 86.

68	$2304 = 4(576) = \underline{(4)(9)(\quad\quad)}$	$(4)(9)(64)$
	$= (2)(2)(3)(3)(8)(8)$	
69	$= \underline{((\quad)(\quad)(\quad))^2}$	$((2)(3)(8))^2$
70	Hence, $\sqrt{2304} = (2)(3)(8) = \underline{\quad\quad}$	48
71	$676 = \underline{4(\quad\quad)}$	$4(169)$
72	$= \underline{(2)(2)(13)(\quad)}$	$(2)(2)(13)(13)$
73	$= \underline{(\quad\quad)^2}$	$((2)(13))^2$
74	$\sqrt{676} = \underline{(2)(\quad)} = 26$	$(2)(13)$
75	$1225 = 5(245) = \underline{(5)(\quad)(\quad)}$	$(5)(5)(49)$
76	$= (5)(5)\underline{(\quad)(\quad)}$	$(7)(7)$
77	$= \underline{(\quad\quad)^2}$	$((5)(7))^2$
78	$-\sqrt{1225} = -(5)(7) = \underline{\quad\quad}$	-35
79	$11.56 = (1156)(0.01) = 4\underline{(\quad\quad)}(0.1)(0.1)$	(289)
80	$= (2)(2)\underline{(17)(\quad)}(0.1)(0.1)$	$(17)(17)$
81	$= \underline{(\quad\quad)^2}$	$((2)(17)(0.1))^2$
82	$\sqrt{11.56} = (34)(0.1) = \underline{\quad\quad}$	3.4
83	$0.0256 = (256)(0.0001) = \underline{4(\quad\quad)(0.01)(\quad\quad)}$	$4(64)(0.01)(0.01)$
84	$= (2)(2)(8)(8)(0.01)(0.01) = \underline{(\quad,\quad)^2}$	$((2)(8)(0.01))^2$
85	$\sqrt{0.0256} = \underline{(\quad)(0.01)} = 0.16$	$(16)(0.01)$

86	Is $\sqrt{x^2} + 2 = 1$ true for some value of x? [A] yes, if x is -1. [B] no	

Remember that $\sqrt{x^2} = |x|$ for all real numbers. Therefore, $\sqrt{x^2}$ is always non-negative. Any non-negative number added to 2 is greater than 1. [B] is correct.

So far, we have been talking about squares and square roots. Now let us look at some cubes of numbers.

87	If $x = 2$, then $x^3 =$ ______.	8
88	If $x = -4$, then $x^3 =$ ______.	-64
89	If $x = -0.3$, then $x^3 =$ ______.	-0.027
90	If $x = \frac{1}{2}a$, then $x^3 =$ ______.	$\frac{1}{8}a^3$
91	If $x = 5n^2$, then $x^3 =$ ______.	$125n^6$

Now we examine some equations about x^3.

92	The truth set of $x^3 = 8$ is { ____ }.	{2}
93	The truth set of $x^3 = -64$ is ______.	{-4}
94	The truth set of $x^3 = -0.027$ is ______.	{-0.3}
95	The truth set of $x^3 = \frac{8}{27}$ is ______.	$\{\frac{2}{3}\}$
96	The truth set of $x^3 = -125$ is ______.	{-5}

When we find the truth set of $x^3 = 8$, we are finding a number whose cube is 8. There is only one such real number, 2, and we say that 2 is the cube root of 8.

In general, if a and b are real numbers such that $a^3 = b$, then a is the cube root of b.

97	Thus, 3 is the cube root of 27 because $3^3 =$ ______.	27
98	-5 is the cube root of ______ because $(-5)^3 = -125$.	-125
99	______ is the cube root of 64 because $4^3 = 64$.	4

If $a^3 = b$, we write $a = \sqrt[3]{b}$ and read it "a is the cube root of b".

100	$\sqrt[3]{64} = \sqrt[3]{(4)^3} =$ ______	4
101	$\sqrt[3]{-125} = \sqrt[3]{(-5)^3} =$ ______	-5
102	$\sqrt[3]{x^3} =$ ______	x
103	$\sqrt[3]{(-13)^3} =$ ______	-13
104	$\sqrt[3]{0.008} = \sqrt[3]{(0.2)^3} =$ ______	0.2
105	$\sqrt[3]{1000} = \underline{\sqrt[3]{(\quad)^3}} =$ ______	$\sqrt[3]{(10)^3}$, 10
106	$\sqrt[3]{\frac{27}{1000}} = \sqrt[3]{(\frac{3}{10})^3} =$ ______	$\frac{3}{10}$
107	$\sqrt[3]{(x-3y)^3} =$ ______	x - 3y

108	The square of any non-zero number is ______ (positive, negative) regardless of whether the number is positive or negative.	positive
109	Therefore, it is not possible to find the square root of a ______ number.	negative
110	However, while the cube of a positive number is positive, the cube of a negative number is ______.	negative
111	Therefore, it <u>is</u> possible to take the cube ______ of a negative number.	root
112	The cube root of a negative number is ______ (positive, negative).	negative

The preceding observations are certainly correct within the framework of the real numbers. Sometime in your study of mathematics you will find that if we extend still further the kinds of numbers we use, we can insure that negative numbers will have square roots too, and that every non-zero number will have three cube roots.

15-2. Irrational Numbers

Thus far you have found simpler names for square roots of rational numbers for which either the square root is obvious or for which the square root may be found by using prime factorization. Not all numbers have this property. $\sqrt{2}$, for example, certainly has no obvious simpler name.

Following are some examples of the types of square roots we have found.

1 $\sqrt{16} =$ ______ — 4

2 $\sqrt{\frac{4}{9}} =$ ______ — $\frac{2}{3}$

3 $\sqrt{y^2} =$ ______ — $|y|$

If we were given an expression such as $\sqrt{\frac{4}{9}} + 1$, we usually would not leave it in this form, since we can write a simpler name for it.

4 $\sqrt{\frac{4}{9}}$ is the rational number ______. — $\frac{2}{3}$

Thus, $\sqrt{\frac{4}{9}} + 1 = \frac{2}{3} + 1$

5 $= \frac{\square}{3}$. — $\frac{5}{3}$

6 $\sqrt{\frac{1}{4}}$, $\sqrt{0.36}$, $\sqrt{256}$ are all ______ numbers. — rational

7 $\sqrt{\frac{9}{25}} + \sqrt{\frac{1}{9}} + 1 =$ ___ + ___ + 1 — $\frac{3}{5} + \frac{1}{3} + 1$

8 $=$ ______. — $\frac{29}{15}$

9 $\sqrt{\frac{25}{16}} - \sqrt{\frac{9}{4}} =$ ______. — $-\frac{1}{4}$

Our entire discussion thus far has been based on an underlying assumption: For any non-negative real number a there is exactly one non-negative number whose square is a. In other words, if $a \geq 0$, then $\sqrt{a}$ is a unique real number.

This assumption cannot be proved on the basis of the properties of the real numbers listed in Chapter 10. It is a consequence of the further property noted there but not discussed.

Let us consider the square root of the number 2. We showed earlier how $\sqrt{2}$ can be located on the number line. In order to discuss more precisely where it is located, we need to show first that if a and b are non-negative numbers such that $a < b$, then $\sqrt{a} < \sqrt{b}$.

Frame		Answer
	We have proved that if a, b, c, and d are positive	
10	numbers and if $a < b$ and $c < d$, then $ac <$ _____.	$ac < bd$
	Thus, if $\sqrt{a} < \sqrt{b}$,	
	then $\sqrt{a} \cdot \sqrt{a} < \sqrt{b} \cdot \sqrt{b}$,	
11	or a _____.	$a < b$
	Also, if $\sqrt{a} > \sqrt{b}$	
12	then a _____.	$a > b$
13	From the _____ property of equality,	multiplication
	if $\sqrt{a} = \sqrt{b}$	
14	then a _____.	$a = b$

On the basis of all of this, it is easy to show by an indirect proof that:

If a and b are non-negative real numbers such that $a < b$, then $\sqrt{a} < \sqrt{b}$.

Frame		Answer
	Proof:	
	From the comparison property it must be true that	
15	either $\sqrt{a} < \sqrt{b}$, $\sqrt{a} = \sqrt{b}$, or _____.	$\sqrt{a} > \sqrt{b}$
	Suppose $\sqrt{a} = \sqrt{b}$, then $a = b$. This contradicts the assumption that $a < b$, so it cannot be true that	
16	$\sqrt{a} =$ _____.	$\sqrt{a} = \sqrt{b}$
	Suppose $\sqrt{a} > \sqrt{b}$. Then $a > b$. This contradicts the assumption that $a < b$, so it cannot be true that	
17	$\sqrt{a}$ _____.	$\sqrt{a} > \sqrt{b}$
	We conclude: If a and b are non-negative real	
18	numbers such that $a < b$, then $\sqrt{a}$ _____.	$\sqrt{a} < \sqrt{b}$

We can use this argument to help locate $\sqrt{2}$ on the number line. We can reason as follows:

Frame		Answer
19	$1^2 < 2 < 2^2$, hence $1 < \sqrt{2} <$ _____.	2
20	$(1.4)^2 < 2 < (1.5)^2$, hence, ___ $< \sqrt{2} <$ ___.	$1.4 < \sqrt{2} < 1.5$
21	$(1.41)^2 < 2 < (1.42)^2$, hence, _____.	$1.41 < \sqrt{2} < 1.42$

Since $1.41 < \sqrt{2}$ and $\sqrt{2} < 1.42$, we see that $\sqrt{2}$ lies between the rational numbers 1.41 and 1.42 on the number line.

22	Every point on the number line corresponds to a ______ number.	real
23	A rational number is a real number which can be named by a fraction in which the numerator is a ______ number and the denominator a whole number different from 0.	whole
24	$\frac{3}{5}$ is a ______ number, as is $\frac{5}{3}$.	rational
25	4 is also a rational number, since $4 = \frac{\square}{\square}$.	$\frac{4}{1}$, or $\frac{8}{2}$, etc.
26	1.42 is a rational number since $1.42 = \frac{\square}{100}$.	$\frac{142}{100}$
27	A real number which is not a rational number is called an ______ number.	irrational

We have seen how to locate $\sqrt{2}$ approximately on the number line. We have assumed that $\sqrt{2}$ is not a rational number, and now we shall proceed to prove that $\sqrt{2}$ is irrational. This we shall state as:

Theorem 15-2. $\sqrt{2}$ is irrational.

Before we begin the actual proof, let us review some ideas which are useful in this proof.

28	If an integer p is even, we can find an integer n such that p = ______.	p = 2n
29	For any integer a, if a^2 is even, then a is ______.	even
30	Thus, if $a^2 = 2n$, where n is a positive integer, then a is ______.	even
31	That is, there is an ______ q such that $a = 2q$.	integer

32	To show that $\sqrt{2}$ ______ (is, is not) rational, we shall assume	is not
33	that the reverse is true; that is, $\sqrt{2}$ is ______.	rational
34	We shall then show that this assumption leads to a ______ (true, false) conclusion.	false
35	An assumption which leads to a false conclusion must itself be ______.	false

If the assumption that $\sqrt{2}$ is rational leads to a
36 false conclusion, then it is false that $\sqrt{2}$ is ______. rational

If it is false that $\sqrt{2}$ is rational, then $\sqrt{2}$ must
37 be ______. irrational

Now we prove

Theorem 15-2. $\sqrt{2}$ is irrational

as follows:

We use the indirect method of proof, assuming that $\sqrt{2}$
38 is a ______ number. rational

Then there are positive integers a and b, with $b \neq 0$, such that

$$\frac{a}{b} = \sqrt{2}$$

and such that a and b have no factor in common.
39 (If there had been a common ______, we could have factor
removed it.) In particular, a and b are not both even.

$$\left(\frac{a}{b}\right)^2 = (\sqrt{2})^2 = 2$$

40 $$\frac{a^2}{b^2} = \_\_\_\_\_\_$$ 2

41 $$a^2 = 2b^2,$$ by the ______ property of equality. multiplication

Thus, a^2 is even. Consequently, a is also even.
Since a is even, there is an integer c such that

42 $$\_\_\_\_\_\_ = 2c$$ $a = 2c$

Since $a^2 = 2b^2$,

43 then $(2c)^2 = \_\_\_\_\_\_$. $2b^2$

$$4c^2 = 2b^2$$

44 $$\_\_\_\_\_\_ = b^2$$ $2c^2$

Since $2c^2 = b^2$, b^2 is even.
45 Consequently, b is also ______. even

Item		Answer
	In Items 42 and 45 we have shown that the assumption that $\sqrt{2}$ is rational leads to the conclusion that a and b are both even.	
46	We asserted earlier that a and b are not both ____.	even
47	Hence, the conclusion that a and b are both even is ____ (true, false).	false
48	Thus, the assumption that $(\frac{a}{b})^2 = 2$ for some integers a and b is ____ (true, false) because it leads to a false conclusion.	false
49	We have shown: $\sqrt{2}$ cannot be expressed as an indicated ____ of two integers.	quotient
50	Therefore, $\sqrt{2}$ is ____.	irrational

Notice an interesting difference between a proof by contradiction, such as we have just done, and other types of proof which you have seen during this course. In the direct proof, there is a specific fact which you are trying to establish, and you proceed to work with whatever facts you are given and with the properties of the real numbers until the fact you are seeking is before you. You concentrate on creating the statement you desire from statements which you have assumed to be true.

In a proof by contradiction (indirect proof), on the other hand, you add to the list of things with which you work the <u>denial</u> of what you want to prove, and then keep deriving results until a contradiction appears. This contradiction proves that you made a mistake in denying what you wanted to show, and thus that what you wanted to show must have been true all along.

Item		Answer
	Let us prove that $\sqrt{2} + 3$ is irrational.	
	Proof:	
51	Assume $\sqrt{2} + 3$ is a ____ number.	rational
52	Then $\sqrt{2} + (3 - \_\_\_)$, or $\sqrt{2}$, is rational, since the set of rational numbers is closed under the operation	(3 - 3)
53	of ____.	addition
54	But $\sqrt{2}$ is <u>not</u> rational. Thus, we have a ____, and	contradiction
55	our assumption that $\sqrt{2} + 3$ is rational is ____ (true, false).	false
	Hence, $\sqrt{2} + 3$ is irrational.	

We have proved that the real numbers $\sqrt{2}$ and $\sqrt{2} + 3$ are not rational numbers. This shows that not every real number is a rational number. As a matter of fact, there are many numbers which can be proved to be irrational by proofs similar to those which we have shown.

Try to prove for yourself that $\frac{1}{2}\sqrt{2} - 1$ is irrational. As the first step of the proof, we assume that $\frac{1}{2}\sqrt{2} - 1$ is ______. The complete proof is on page x. Turn to this page only after you have written your own proof.	rational

57	Try to prove for yourself that $\sqrt{3}$ is irrational. As a first step we assume that there are two integers a, b such that $\frac{a}{b} =$ ______, and that a and b have no factor in common. The complete proof is on page x. Turn to this page after you have written your own proof.	$\sqrt{3}$

15-3. Simplification of Radicals

The fact that a positive number n has exactly one positive square root must be kept in mind now, as we investigate some techniques which help us to simplify expressions involving radicals.

Suppose that we consider the product of two square roots, say $\sqrt{25}$ and $\sqrt{4}$.	
$\sqrt{25} =$ ______; and $\sqrt{4} = 2$.	5
On the other hand, $\sqrt{25 \cdot 4} = \sqrt{100} =$ ______.	10
This enables us to see that $\sqrt{25} \cdot \sqrt{4} = \sqrt{\quad \cdot 4}$.	$\sqrt{25 \cdot 4}$

122

Now, let us turn to the product of $\sqrt{2}$ and $\sqrt{3}$ and see if we can write this as a simpler expression.

4 We suspect that $\sqrt{2} \cdot$ ______ $= \sqrt{2 \cdot 3} = \sqrt{6}$. $\sqrt{3}$

To verify that $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$ is true, notice that $\sqrt{6}$

5 is a positive number whose square is ______; that is, $(\sqrt{6})^2 = 6$. 6

6 But $(\sqrt{2} \cdot \sqrt{3})^2 = \underline{(\sqrt{2})^2 \cdot (\quad)^2}$. $(\sqrt{2})^2 \cdot (\sqrt{3})^2$

7 $= 2 \cdot$ ______ 3

8 ______. 6

So $\sqrt{6}$ and $\sqrt{2} \cdot \sqrt{3}$ are both positive numbers whose

9 square is ______. 6

We know, however, that there is only <u>one</u> positive number

10 whose square is ______. 6

11 Hence, $\sqrt{2} \cdot \sqrt{3} = \sqrt{\quad}$______. $\sqrt{6}$

These examples suggest that we can prove:

<u>Theorem 15-3</u>. For any positive numbers a and b,

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}.$$

12 Proof: $(\sqrt{a} \cdot \sqrt{b})^2 = (\sqrt{a} \cdot \sqrt{b})(\underline{\quad\quad})$ $(\sqrt{a} \cdot \sqrt{b})$

$= (\sqrt{a})^2(\sqrt{b})^2$

13 $=$ ______. ab

14 Hence, $\sqrt{a} \cdot \sqrt{b}$ is a number whose ______ is ab. square

$\sqrt{a} \cdot \sqrt{b}$ is a positive number, since both $\sqrt{a}$ and

15 $\sqrt{b}$ are ______. positive

By definition, $\sqrt{ab}$ is also a positive number whose

16 square is ______. ab

17 But the positive number ab has only ______ positive square root. one

Hence, $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$.

18 Which of the following are true sentences?

M. $\sqrt{5} \cdot \sqrt{6} = \sqrt{30}$

N. $\sqrt{2} \cdot \sqrt{7} = \sqrt{14}$

O. $\sqrt{5} \cdot \sqrt{0} = 0$

P. $\sqrt{11} \cdot \sqrt{11} = 11$

Q. $\sqrt{5} \cdot \sqrt{4 \cdot 2} = \sqrt{\text{[illegible]}0}$

[A] all | [C] all except Q

[B] all except M and N | [D] all except O and Q

Look at Q:

$$\sqrt{5} \cdot \sqrt{4 \cdot 2} = \sqrt{5(4 \cdot 2)}$$
$$= \sqrt{40}$$

Hence, Q is not a true sentence. The same sort of reasoning will show that each of the other sentences <u>is</u> true. Hence, [C] is the correct choice.

Write each indicated product as a rational number if possible. Otherwise write it as a single radical.

19 $\sqrt{3} \cdot \sqrt{3}$ ______ 3

20 $\sqrt{3} \cdot \sqrt{5}$ ______ $\sqrt{15}$

21 $\sqrt{2} \cdot \sqrt{8}$ ______ 4

22 $\sqrt{2} \cdot \sqrt{11} \cdot \sqrt{3}$ ______ $\sqrt{66}$

23 $\sqrt{3} \cdot \sqrt{12}$ ______ 6

24 $\sqrt{0} \cdot \sqrt{7}$ ______ 0

We have seen that for all non-negative numbers a and b,

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}.$$

We can often use this fact to separate a single square root into the product of two square roots.

25 $\sqrt{6} = \sqrt{2 \cdot 3}$ = ______ $\sqrt{2} \cdot \sqrt{3}$

26 $\sqrt{12} = \sqrt{4} \cdot$ ______ = ______ $\sqrt{4} \cdot \sqrt{3}$, $2\sqrt{3}$

27 We could also write $\sqrt{12} = \sqrt{6} \cdot$ ______. $\sqrt{6} \cdot \sqrt{2}$

However, $\sqrt{4} \cdot \sqrt{3}$ has the advantage that $\sqrt{4}$ is a

28 ______ number. We will often find $2\sqrt{3}$ simpler to use than $\sqrt{12}$. rational

29	$\sqrt{50} = \sqrt{} \cdot \sqrt{2}$	$\sqrt{25} \cdot \sqrt{2}$
30	$= \underline{}.$	$5\sqrt{2}$
31	$\sqrt{27} = \underline{\sqrt{} \cdot \sqrt{}}$	$\sqrt{9}\sqrt{3}$
32	$= \underline{}.$	$3\sqrt{3}$
33	$5\sqrt{18} = \underline{5\sqrt{} \cdot \sqrt{}}$	$5\sqrt{9} \cdot \sqrt{2}$
34	$= 5(3)\sqrt{2} = \underline{}.$	$15\sqrt{2}$
35	$2\sqrt{6} \cdot 3\sqrt{15} = (2\sqrt{2} \cdot \sqrt{3})\underline{(3\sqrt{} \cdot \sqrt{})}$	$(3\sqrt{3} \cdot \sqrt{5})$
36	$= \underline{6\sqrt{2}(\sqrt{})^2\sqrt{5}}$	$6\sqrt{2}(\sqrt{3})^2\sqrt{5}$
37	$= 6(3)\sqrt{2} \cdot \sqrt{5} = \underline{}.$	$18\sqrt{10}$

Write each of the following in simpler form, if possible, making the number under the radical sign the smallest whole number possible:

38	$\sqrt{250} = \underline{}$	$5\sqrt{10}$
39	$\sqrt{80} = \underline{}$	$4\sqrt{5}$
40	$\sqrt{16} = \underline{}$	4
41	$\sqrt{192} = \underline{}$	$8\sqrt{3}$
42	$7\sqrt{28} = \underline{}$	$14\sqrt{7}$
43	$3\sqrt{52} = \underline{}$	$6\sqrt{13}$
44	$5\sqrt{13} = \underline{}$	$5\sqrt{13}$
45	$\sqrt{5} \cdot \sqrt{20} = \underline{}$	10

If you were asked to find the simplest form of $\sqrt{108}$ you might notice that

46	$\sqrt{108} = \underline{\sqrt{} \cdot \sqrt{3}}$ $= 6\sqrt{3}.$	$\sqrt{36} \cdot \sqrt{3}$

If you had not noticed this, you might have seen that

47	$\sqrt{108} = \underline{\sqrt{9} \cdot \sqrt{}}$	$\sqrt{9}\sqrt{12}$
	$= 3\sqrt{12}$	
48	$= \underline{}.$	$6\sqrt{3}$

Again, you might have used the prime factorization of 108 to write

$$\sqrt{108} = \sqrt{2^2 \cdot 3^3}$$
$$= \sqrt{2^2 \cdot 3^2 \cdot 3}$$
49 $$= 2 \cdot 3\sqrt{\quad}$$ $2 \cdot 3\sqrt{3}$
$$= 6\sqrt{3}$$

Notice that in the last method we group the highest even powers of the factors.

Simplify. Answers are on page xi.

50. $4\sqrt{121} =$ ______
51. $\sqrt{18} \cdot \sqrt{27} =$ ______
52. $(6\sqrt{2})(3\sqrt{14}) =$ ______
53. $(\sqrt{7})(2\sqrt{11}) =$ ______
54. $(3\sqrt{38})(2\sqrt{57}) =$ ______
55. $\sqrt{9 + 16} =$ ______
56. $\sqrt{9 \cdot 16} =$ ______
57. $\sqrt{9} + \sqrt{16} =$ ______
58. $\sqrt{9} - \sqrt{16} =$ ______
59. $\sqrt{916} =$ ______
60. $\sqrt{9 + 25} =$ ______

61 $\sqrt{2}(\sqrt{3} + \sqrt{8}) = \sqrt{2} \cdot \sqrt{3} + \sqrt{2} \cdot$ ______ $\sqrt{2} \cdot \sqrt{8}$
62 $= \sqrt{\quad} + \sqrt{16}$ $\sqrt{6} + \sqrt{16}$
63 $=$ ______ $\sqrt{6} + 4$
64 $\sqrt{3}(\sqrt{6} + \sqrt{2}) = \sqrt{\quad} + \sqrt{\quad}$ $\sqrt{18} + \sqrt{6}$
65 $=$ ______ $3\sqrt{2} + \sqrt{6}$
66 $2\sqrt{5}(\sqrt{2} + 3\sqrt{5}) =$ ______ $2\sqrt{10} + 30$
67 $3\sqrt{7}(\sqrt{21} + 1) =$ ______ $21\sqrt{3} + 3\sqrt{7}$
68 $5\sqrt{3}(\sqrt{8} - \sqrt{6}) =$ ______ $10\sqrt{6} - 15\sqrt{2}$
69 $\sqrt{15}(\sqrt{3} - \sqrt{5}) =$ ______ $3\sqrt{5} - 5\sqrt{3}$

70 $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) = \sqrt{3}(\quad) + \sqrt{2}(\sqrt{3} - \sqrt{2})$ $(\sqrt{3} - \sqrt{2})$
71 $= 3 - \sqrt{6} +$ ______ $- 2$ $\sqrt{6}$
72 $= 3 - 2 =$ ______ 1
73 $(\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3}) = \sqrt{2}(\quad) +$ ______ $(\sqrt{2} - \sqrt{3})$ $(\sqrt{2} - \sqrt{3}), \sqrt{3}$
74 $= 2 - \sqrt{6} +$ ______ $2 - \sqrt{6} + \sqrt{6} - 3$
75 $= 2 - 3 =$ ______ -1
76 $(\sqrt{3} + \sqrt{5})(\sqrt{3} - \sqrt{5}) =$ ______ -2

$(\sqrt{2} + \sqrt{3})^2 = (\sqrt{2} + \sqrt{3})(\sqrt{2} + \sqrt{3})$

*77 $= \sqrt{2}(\sqrt{2} + \sqrt{3}) + \sqrt{3}(\underline{\qquad})$ — $\sqrt{3}(\sqrt{2} + \sqrt{3})$

*78 $= \underline{2 + \quad + \quad + 3}$ — $2 + \sqrt{6} + \sqrt{6} + 3$

*79 $= \underline{\quad + 2\sqrt{6}}$ — $5 + 2\sqrt{6}$

*80 $(\sqrt{5} + \sqrt{3})^2 = \underline{\qquad}$ — $8 + 2\sqrt{15}$

*81 $(\sqrt{5} - \sqrt{2})^2 = \underline{\qquad}$ — $7 - 2\sqrt{10}$

82 $\sqrt{x^2} = \underline{\qquad}$ — $|x|$

83 $\sqrt{9x^2} = \underline{\qquad}$ — $3|x|$

84 $\sqrt{3x^2} = \underline{\sqrt{x^2} \cdot \sqrt{\quad}} = \underline{\qquad}$ — $\sqrt{x^2} \cdot \sqrt{3}$, $|x|\sqrt{3}$

In each of the above items, recall that since x^2

85 is non-______ for all values of x, then — non-negative

86 ______ $= |x|$. — $\sqrt{x^2} = |x|$

87 x^3, on the other hand, is negative if x is ______. — negative

Hence, the domain of the variable in $\sqrt{x^3}$ is the set

88 of non-______ real numbers. — non-negative

Thus, for any number in the domain of $\sqrt{x^3}$, we see

89 that $\sqrt{x^3} = \sqrt{x^2} \cdot \sqrt{x} = $ ______. [Note that since x is non-negative we do not need to write $|x|\sqrt{x}$.] — $x\sqrt{x}$

90 $\sqrt{24x^2} = \underline{\sqrt{4x^2} \cdot \sqrt{\quad}}$ — $\sqrt{4x^2} \cdot \sqrt{6}$

91 $=$ ______, where x is any real number. — $2|x|\sqrt{6}$

92 $\sqrt{24x^3} = \underline{\sqrt{4x^2} \cdot \sqrt{\quad}}$ — $\sqrt{4x^2} \cdot \sqrt{6x}$

93 $=$ ______, where x is non-negative. — $2x\sqrt{6x}$

$\sqrt{24x^4} = \sqrt{4x^4} \cdot \sqrt{6}$

94 $=$ ______, where x is any real number. [since x^2 is non-negative] — $2x^2\sqrt{6}$

95 $\sqrt{24x^5} = \underline{\sqrt{4x^4} \cdot \sqrt{\quad}}$ — $\sqrt{4x^4} \cdot \sqrt{6x}$

96 $=$ ______, where x is non-negative. — $2x^2\sqrt{6x}$

In the preceding examples we have been careful to state the set of values of the variable for which the radical has meaning. Where the domain is the set of all real numbers, $\sqrt{x^2}$, which is always non-negative, is $|x|$. If the domain is the set of non-negative real numbers, then x is non-negative and we can write $\sqrt{x^2}$ as x.

	When we write $\sqrt{a}$ we know that the domain of a must be restricted to non-negative real numbers. Consider $\sqrt{3x - 1}$.	
97	Is 0 in the domain of x ? ______ (Hint: Find the value of $3x - 1$ when x is 0.)	no
98	Is $\frac{1}{2}$ in the domain of x ? ______	yes
99	Is $\frac{1}{3}$ in the domain of x ? ______	yes
100	Is -1 in the domain of x ? ______	no
101	The ______ of x is the set of real numbers which	domain
102	are greater than or equal to ______.	$\frac{1}{3}$

103	Since $x^4 + x^2 = x^2(\quad)$, we see readily that	$x^2(x^2 + 1)$
104	$\sqrt{x^4 + x^2} = \sqrt{\quad}\sqrt{x^2 + 1}$	$\sqrt{x^2}$
	$= \|x\|\sqrt{x^2 + 1}$	
	In $\sqrt{x^4 + x^2}$, the domain of x is the set of all	
105	______ numbers, since x^2 and x^4 are always non-negative.	real

106	In $\sqrt{y} \cdot \sqrt{y^3}$, the domain is the set of ______ real numbers.	non-negative
107	Hence, $\sqrt{y} \cdot \sqrt{y^3} = \sqrt{y^4} =$ ______, where y is non-negative.	y^2
108	In $\sqrt{72(x - 1)^2}$, the domain is the set of ______ real numbers.	all
109	Hence, $\sqrt{72(x - 1)^2} = \sqrt{36(x - 1)^2} \cdot \sqrt{2} =$ ______.	$6\|x - 1\|\sqrt{2}$

Simplify, indicating the domain of the variable in each problem in which it is not the set of all real numbers. Answers on page xi.

110. $\sqrt{2} \cdot \sqrt{x} =$ ______	115. $\sqrt{32a^4} =$ ______
111. $\sqrt{3} \cdot \sqrt{x^3} =$ ______	116. $\sqrt{625x^2} =$ ______
112. $\sqrt{45n^2} =$ ______	117. $\sqrt{5x^6} =$ ______
113. $(2\sqrt{36})(5\sqrt{6x}) =$ ______	118. $\sqrt{(x^3)(x)} =$ ______
114. $\sqrt{75y^7} =$ ______	119. $\sqrt{600x} \cdot \sqrt{5000} =$ ______

* We have stated that any positive real number has exactly one positive square root. It can be shown that any real number has exactly one real cube root.

Using this fact, we can also prove: $\sqrt[3]{a}\,\sqrt[3]{b} = \sqrt[3]{ab}$ for all real numbers a and b. Try to complete the proof for yourself. Then use Items 120 to 124 as a help or as a check.

*120	$(\sqrt[3]{a}\,\sqrt[3]{b})^3 = (\sqrt[3]{a}\,\sqrt[3]{b})(\sqrt[3]{a}\,\sqrt[3]{b})(\sqrt[3]{a}\,\sqrt[3]{b})$ $= (\sqrt[3]{a})^3(\sqrt[3]{})^3$ $= ab$	$(\sqrt[3]{b})^3$
*121	Hence, $\sqrt[3]{a}\,\sqrt[3]{b}$ is a number whose cube is ______.	ab
*122	$\sqrt[3]{ab}$ is also, by definition, a number whose cube is ___.	ab
*123	But the real number ab has only ______ (how many) cube root.	one
*124	Hence, $\sqrt[3]{a}\,\sqrt[3]{b} = \sqrt[3]{}$ ______.	$\sqrt[3]{ab}$

	Since for any real number x, $\sqrt[3]{x^3} = x$, then:	
*125	$\sqrt[3]{24x^2} = \sqrt[3]{8} \cdot \sqrt[3]{3x^2} =$ ______	$2\sqrt[3]{3x^2}$
*126	$\sqrt[3]{24x^3} = \sqrt[3]{8x^3} \cdot \sqrt[3]{3} =$ ______	$2x\sqrt[3]{3}$
*127	$\sqrt[3]{32a^4} =$ ______	$2a\sqrt[3]{4a}$
*128	$\sqrt[3]{27a^2} =$ ______	$3\sqrt[3]{a^2}$
*129	$\sqrt[3]{-27b^3} =$ ______	$-3b$
*130	$\sqrt[3]{-27c^5} =$ ______	$-3c\sqrt[3]{c^2}$

15-4. Simplification of Radicals Involving Fractions

We have simplified certain radicals which contained integers and powers of variables under the radical sign. Now we shall look at radical expressions involving fractions.

	Examine $\sqrt{\frac{49}{64}}$.	
1	$\sqrt{\frac{49}{64}} = \sqrt{\left(\frac{7}{8}\right)^2} =$ ______.	$\frac{7}{8}$
2	On the other hand, $\sqrt{49} = 7$, $\sqrt{64} =$ ______,	8
3	so that $\frac{\sqrt{49}}{\sqrt{64}} = \frac{7}{\square} = \sqrt{\frac{49}{64}}$.	$\frac{7}{8}$

This example suggests a generalization, which we state as a theorem.

Theorem 15-4. If $a \geq 0$ and $b > 0$, then

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} .$$

4	Proof: $\left(\frac{\sqrt{a}}{\sqrt{b}}\right)^2 = \frac{\sqrt{a}}{\sqrt{b}} \cdot \frac{\square}{\square}$.	$\frac{\sqrt{a}}{\sqrt{b}} \cdot \frac{\sqrt{a}}{\sqrt{b}}$
	$= \frac{(\sqrt{a})^2}{(\sqrt{b})^2}$	
5	$=$ ______	$\frac{a}{b}$
6	Thus, $\frac{\sqrt{a}}{\sqrt{b}}$ is a ______ number whose square is $\frac{a}{b}$.	positive
7	By definition, $\sqrt{\frac{a}{b}}$ is also a positive number whose square is ______.	$\frac{a}{b}$
8	But the positive number $\frac{a}{b}$ has only ______ positive square root.	one
	Hence, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.	

	Simplify:	
9	$\sqrt{\frac{12}{25}} = \frac{\sqrt{12}}{\sqrt{25}} =$ ______ .	$\frac{2\sqrt{3}}{5}$, or $\frac{2}{5}\sqrt{3}$
10	$\sqrt{\frac{16}{25}} =$ ______	$\frac{4}{5}$
11	$\sqrt{1 + \frac{24}{25}} = \sqrt{\frac{\square}{25}} =$ ______	$\sqrt{\frac{49}{25}} = \frac{7}{5}$

We can use Theorem 15-4 to simplify an expression such as $\sqrt{\frac{15}{5x^2}}$ $(x \neq 0)$.

12 $\sqrt{\frac{15}{5x^2}} = \sqrt{\frac{\square}{x^2}}$ — $\sqrt{\frac{3}{x^2}}$

$= \frac{\sqrt{3}}{\sqrt{x^2}}$, by Theorem 15-4

13 $= \frac{\sqrt{3}}{\square}$. — $\frac{\sqrt{3}}{|x|}$

Simplify:

14 $\sqrt{\frac{x^2}{9}} =$ ______ — $\frac{|x|}{3}$

15 $\sqrt{\frac{5w^2}{y^2}} =$ ______, $y \neq 0$. — $\frac{|w|\sqrt{5}}{|y|}$, or $\left|\frac{w}{y}\right|\sqrt{5}$

16 Which of the following statements are true for all allowable values of the variable?

P. $\sqrt{\frac{3x^2}{75}} = \frac{x}{5}$ R. $\frac{\sqrt{a^5}}{\sqrt{9a^3}} = \frac{a}{3}$

Q. $\sqrt{\frac{3x^2}{75}} = \frac{|x|}{5}$ S. $\frac{\sqrt{a^5}}{\sqrt{9a^3}} = \frac{|a|}{3}$

[A] all are true

[B] all but P are true

[C] Q and S are true, but P and R are false

[B] is correct. If you were not sure, or if you made the wrong choice, complete Items 17-21. If not, go to Item 22.

If x is any real number, x^2 is non-negative.

In $\sqrt{\frac{3x^2}{75}}$, the domain of x is the set of all

17 ______ numbers. — real

18 Hence, $\sqrt{\frac{3x^2}{75}} =$ ______. — $\frac{|x|}{5}$

In $\frac{\sqrt{a^5}}{\sqrt{9a^3}}$, the domain of a is the set of

19 all ______ numbers. — positive

If a is any positive number, then $\sqrt{a^2} = a$ is a true statement.

Hence, we may write:

$$\frac{\sqrt{a^5}}{\sqrt{9a^3}} = \sqrt{\frac{a^5}{9a^3}}$$

20 $\sqrt{\frac{a^2}{9}} =$ ______ $\frac{a}{3}$

21 However, if a is positive, $|a| =$ ______, and it is also correct to say: a

$$\frac{\sqrt{a^5}}{\sqrt{9a^3}} = \frac{|a|}{3}$$

The absolute value symbol is not incorrect here, but it is not necessary.

Simplify, indicating the domain of the variable when it is restricted. Check your answers with those on page xi.

22. $\sqrt{\frac{3}{25}} =$ ______

23. $\sqrt{\frac{49}{a^2}} =$ ______

24. $\sqrt{\frac{4y}{9y^3}} =$ ______

25. $\sqrt{\frac{6}{27a^2}} =$ ______

26. $\frac{\sqrt{12}}{\sqrt{27}} =$ ______

27. $\sqrt{\frac{35}{4} + \frac{7}{9}} =$ ______

28. $\sqrt{\frac{15}{4} - \frac{5}{6} + \frac{5}{9}} =$ ______

Perform the indicated multiplications and simplify:

29 $\sqrt{\frac{3}{5}} \cdot \sqrt{\frac{10}{49}} = \sqrt{\frac{3 \cdot 10}{5 \cdot 49}} =$ ______ $\frac{\sqrt{6}}{7}$

30 $\sqrt{\frac{m}{11}} \cdot \sqrt{\frac{3m}{44}} =$ ______ $\frac{m\sqrt{3}}{22}$

In the last sentence, the domain of m was the set of

31 all ______ numbers. **non-negative**

32 $\sqrt{\frac{5}{3a}} \cdot \sqrt{\frac{3}{5a}} =$ ______, $a > 0$. $\frac{1}{a}$

33 $\frac{\sqrt{3}}{\sqrt{5}}$ is an indicated ______ of two square roots of integers. — **quotient**

To find a fraction equivalent to $\frac{\sqrt{3}}{\sqrt{5}}$, but with no

34 radical in the numerator, we can use the ______ property of 1 to write: — **multiplication**

35 $$\frac{\sqrt{3}}{\sqrt{5}} = \frac{\sqrt{3}}{\sqrt{5}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{(\sqrt{3})^2}{\square}$$ — $\frac{(\sqrt{3})^2}{\sqrt{15}}$

36 $$= \frac{\square}{\sqrt{15}}$$ — $\frac{3}{\sqrt{15}}$

On the other hand, to obtain a fraction equivalent to $\frac{\sqrt{3}}{\sqrt{5}}$, but with no radical in the <u>denominator</u>, we write:

37 $\frac{\sqrt{3}}{\sqrt{5}} = \frac{\sqrt{3}}{\sqrt{5}}\left(\frac{\sqrt{5}}{\sqrt{5}}\right)$ by the multiplication property of ______. — **1**

$$= \frac{\sqrt{15}}{(\sqrt{5})^2}$$

38 $= $ ______. — $\frac{\sqrt{15}}{5}$

Changing $\frac{\sqrt{3}}{\sqrt{5}}$ into $\frac{3}{\sqrt{15}}$ is called <u>rationalizing</u> the numerator.

39 We ______ the numerator when we obtained the fraction $\frac{3}{\sqrt{15}}$, in which the numerator is rational, as an equivalent fraction for $\frac{\sqrt{3}}{\sqrt{5}}$. — **rationalized**

40 Likewise, changing $\frac{\sqrt{3}}{\sqrt{5}}$ into $\frac{\sqrt{15}}{5}$ is called ______ the <u>denominator</u>. — **rationalizing**

To rationalize the numerator of $\frac{\sqrt{5}}{\sqrt{7}}$, we write:

41 $$\frac{\sqrt{5}}{\sqrt{7}} = \frac{\sqrt{5}}{\sqrt{7}} \cdot \frac{\square}{\square} = \frac{5}{\sqrt{35}}$$ — $\frac{\sqrt{5}}{\sqrt{5}}$

To rationalize the denominator we proceed as follows:

42 $$\frac{\sqrt{5}}{\sqrt{7}} = \frac{\sqrt{5}}{\sqrt{7}} \cdot \frac{\square}{\square}$$ — $\frac{\sqrt{7}}{\sqrt{7}}$

$$= \frac{\sqrt{35}}{7}$$

We now have two ways to simplify a fraction such as $\frac{\sqrt{5}}{\sqrt{7}}$. We may rationalize the numerator, yielding $\frac{5}{\sqrt{35}}$. We may rationalize the denominator, yielding $\frac{\sqrt{35}}{7}$. Which of the simplifications do we prefer? Actually, the answer to this question is dependent on the use we are to make of the simplified form.

Suppose that we wish to find a decimal approximation of $\sqrt{\frac{3}{5}}$, and that we know that $\sqrt{15}$ is approximately 3.873.

43 $\sqrt{\frac{3}{5}} = \frac{\sqrt{15}}{5} = \frac{\square}{5}$. — $\frac{3.873}{5}$

44 $\sqrt{\frac{3}{5}} = \frac{3}{\sqrt{15}} =$ ______ — $\frac{3}{3.873}$

Of the two indicated quotients, $\frac{3.873}{5}$ and $\frac{3}{3.873}$,

45 the easier to compute is ______. — $\frac{3.873}{5}$

For this reason, we often prefer to rationalize

46 the ______. — **denominator**

Rationalize the denominator in each of the following:

47 $\sqrt{\frac{7}{12}} = \frac{\sqrt{7}}{\square}$ — $\frac{\sqrt{7}}{\sqrt{12}}$

$= \frac{\sqrt{7}}{2\sqrt{3}}$

48 $= \frac{\sqrt{7}}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\square}$ — $\frac{\sqrt{3}}{\sqrt{3}}$

49 $= \frac{\sqrt{21}}{\square}$ — $\frac{\sqrt{21}}{6}$

$\sqrt{\frac{3}{2x^2}} = \frac{\sqrt{3}}{\sqrt{2}|x|}$ $(x \neq 0)$

50 $= \frac{\sqrt{3}}{\sqrt{2}|x|} \cdot \frac{\square}{\square}$ — $\frac{\sqrt{2}}{\sqrt{2}}$

51 $=$ ______ — $\frac{\sqrt{6}}{2|x|}$

Rationalize the denominators of each of the following, assuming all variables to be positive.

52 $\sqrt{\frac{1}{3}} =$ ______ — $\frac{\sqrt{3}}{3}$

53 $\sqrt{\frac{1}{27}} =$ ______ — $\frac{\sqrt{3}}{9}$

54 $\sqrt{\frac{9}{50}} =$ ______ — $\frac{3\sqrt{2}}{10}$

55	$\sqrt{\frac{75}{63}}$ = ______	$\frac{5\sqrt{21}}{21}$
56	$\sqrt{\frac{3b}{5}}$ = ______	$\frac{\sqrt{15b}}{5}$
57	$\sqrt{\frac{2a}{7b}}$ = ______	$\frac{\sqrt{14ab}}{7b}$

Perform the indicated operations and rationalize the denominators:

58	$\sqrt{\frac{3}{5}} \cdot \sqrt{\frac{2}{15}}$ = ______	$\frac{\sqrt{2}}{5}$
59	$\sqrt{\frac{2a}{45}} \cdot \sqrt{\frac{a^5}{2}}$ = ______, where a $\underset{(>,\geq)}{\underline{\qquad}}$ 0.	$\frac{a^3\sqrt{5}}{15}$, $a \geq 0$
60	$\frac{\sqrt{6} + \sqrt{24}}{\sqrt{6}} = \frac{\sqrt{6}(\square + \square)}{\sqrt{6}}$ = ______	$\frac{\sqrt{6}(1+2)}{\sqrt{6}}$; 3
61	$\sqrt{\frac{9 + 16}{25}}$ = ______	1
62	$3\sqrt{\frac{1}{3}} \cdot 5\sqrt{\frac{3}{5}}$ = ______	$3\sqrt{5}$

Rationalize the <u>numerators</u> of the following:

63	If $x > 0$, $\frac{\sqrt{x}}{x} = \frac{\sqrt{x}}{x}\left(\frac{\sqrt{x}}{\square}\right)$	$\frac{\sqrt{x}}{\sqrt{x}}$
64	$= \frac{\square}{x\sqrt{x}}$ = ______	$\frac{x}{x\sqrt{x}} = \frac{1}{\sqrt{x}}$
65	If $b > 0$, $\frac{\sqrt{b}}{1 + \sqrt{b}}$ = ______	$\frac{b}{\sqrt{b} + b}$
66	$\frac{3\sqrt{7}}{14\sqrt{9}}$ = ______	$\frac{1}{2\sqrt{7}}$

We can often apply the distributive property in simplifying phrases involving radicals.

	$4\sqrt{3} + 3\sqrt{12} = 4\sqrt{3} + 3(2)\sqrt{3}$	
67	$= 4\sqrt{3}$ + ______	$4\sqrt{3} + 6\sqrt{3}$
68	$= (4 + 6)\sqrt{3}$ = ______	$10\sqrt{3}$
69	Thus, the simple form of $4\sqrt{3} + 3\sqrt{12}$ is ______, since there is no further indicated operation which can be performed.	$10\sqrt{3}$
70	$\sqrt{2} + \sqrt{3}$ is already in simplest ______.	form

71	$\sqrt{12} + \sqrt{50} = 2\sqrt{3} +$ ____. This cannot be further simplified.	$2\sqrt{3} + 5\sqrt{2}$
72	$\sqrt{2} + \sqrt{8} = \sqrt{2} + \square$	$\sqrt{2} + 2\sqrt{2}$
73	$= (1 + 2)\square$	$(1 + 2)\sqrt{2}$
	$= 3\sqrt{2}$	
74	$2\sqrt{12} + 5\sqrt{27} = 4\sqrt{} +$ ____	$4\sqrt{3} + 15\sqrt{3}$
75	$(4 + 15)\sqrt{}$	$(4 + 15)\sqrt{3}$
76	____	$19\sqrt{3}$
	$\sqrt{9a} + \sqrt{4a} = 3\sqrt{a} + 2\sqrt{a}$	
77	____	$5\sqrt{a}$
	$8\sqrt{\frac{1}{2}} - \frac{1}{2}\sqrt{8} = 8(\frac{\sqrt{2}}{2}) - \frac{1}{2}(2\sqrt{2})$	
78	$= 4\sqrt{2} -$ ____	$4\sqrt{2} - \sqrt{2}$
79	$=$ ____	$3\sqrt{2}$

Simplify, if possible. Answers are on page xii.

80. $\sqrt{18} - \sqrt{27}$ ____

81. $\sqrt{\frac{2}{9}} + \sqrt{\frac{2}{5}}$ ____

82. $\sqrt{54} + \frac{1}{2}\sqrt{16} - \sqrt{20} =$ ____

83. $\frac{1}{3}\sqrt{63} + 3\sqrt{7} =$ ____

84. $\frac{1}{4}\sqrt{288} - \frac{1}{6}\sqrt{72} + \frac{1}{\sqrt{24}} =$ ____

To summarize, if we have a sum of different square roots no one of which contains a perfect square factor, then the sum is in simplest form.

	Simplify, assuming a and b are positive numbers.	
85	$\sqrt{9a} + \sqrt{4a} =$ ____	$5\sqrt{a}$
86	$a\sqrt{3a} + 2\sqrt{a^3} =$ ____	$a\sqrt{3a} + 2a\sqrt{a}$, or $a\sqrt{a}(\sqrt{3} + 2)$
87	$\frac{\sqrt{a}}{\sqrt{b}} - \frac{\sqrt{ab}}{b} =$ ____	0

136

In order to find the truth set of

$$2x^2 = 32 \text{ ,}$$

we can form a chain of equivalent equations:

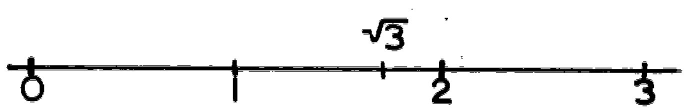
$$2x^2 = 32$$

$$x^2 = 16$$

Frame	Item	Answer
88	$x = 4$ or $x =$ ______	$x = -4$
89	The truth set is ______.	$\{4,-4\}$
90	The truth set of $\frac{1}{3}y^2 = 16$ is ______.	$\{4\sqrt{3},-4\sqrt{3}\}$
91	The equation $(n - 1)^2 = 9$ is equivalent to $n - 1 = 3$ or $n - 1 =$ ______.	$n - 1 = -3$
92	Hence, the truth set of "$(n - 1)^2 = 9$" is ______.	$\{4,-2\}$
93	$3^2 = 9$ and $(-3)^2 =$ ______	9
94	Hence, the truth set of $x^2 = 9$ is ______.	$\{-3,3\}$
95	The truth set of $x^2 = 5$ is ______.	$\{-\sqrt{5},\sqrt{5}\}$
96	The truth set of $\sqrt{x} = 5$ is ______.	$\{25\}$ [(-25) is not in the truth set.]
97	The truth set of $\sqrt{x^2} = 5$ is ______.	$\{-5,5\}$
98	The truth set of $\lvert x\rvert = 5$ is ______.	$\{-5,5\}$

15-5. Approximate Square Roots of Numbers between 1 and 100

We have seen that $\sqrt{3}$ is a real number. Therefore, it can be associated with some point on the number line. Since we know that $1^2 = 1$, and $2^2 = 4$, the point $\sqrt{3}$ must lie somewhere between 1 and 2.



Often we would like to locate the point $\sqrt{3}$ more accurately. In this section we shall learn a method for estimating, or approximating, $\sqrt{a}$ if $1 < a < 100$. How might we estimate $\sqrt{3}$?

Since $1^2 = 1$ and

1 $(\quad)^2 = 3$ and $(\sqrt{3})^2 = 3$

$2^2 = 4$,

2 $\sqrt{3}$ is between ______ and ______. 1, 2

Each of the rational numbers 1 and 2 is called a rational approximation to $\sqrt{3}$.

In fact, 1 and 2 are the nearest integer approximations to

3 ______. $\sqrt{3}$

To get a closer approximation to $\sqrt{3}$, consider the squares:

	$(1.1)^2 = 1.21$	$(1.5)^2 = 2.25$	
	$(1.2)^2 = 1.44$	$(1.6)^2 = 2.56$	
4	$(1.3)^2 = 1.69$	$(1.7)^2 =$ ______	2.89
5	$(1.4)^2 = 1.96$	$(1.8)^2 =$ ______	3.24
		$(1.9)^2 = 3.61$	

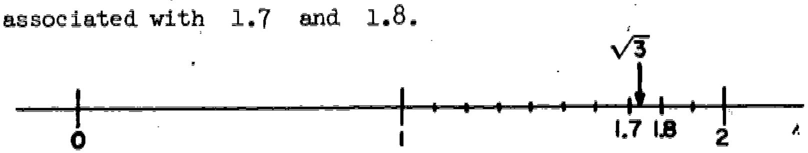
If we look at the squares of 1.1, 1.2, etc., we see that:

6 $(1.7)^2 < 3$ and $(1.8)^2$ ___ 3. $(1.8)^2 > 3$

Closer approximations to $\sqrt{3}$ than the integers 1 and

7 2 are 1.7 and ______. 1.8

This should make it clear that the point associated with $\sqrt{3}$ lies between the points associated with 1.7 and 1.8.



We can indicate that 1.7 and 1.8 are rational approximations (to tenths) of the square root of 3 by using the symbol "<" as follows:

8 $1.7 < \sqrt{3} <$ ______. | $1.7 < \sqrt{3} < 1.8$

We can read this in two ways:

We can say that 1.7 is less than $\sqrt{3}$ and

9 $\sqrt{3}$ is less than ______. | 1.8

10 We can also say that ______ is between 1.7 and 1.8. | $\sqrt{3}$

Computing the squares of the numbers 1.71, 1.72, etc., which are between 1.7 and 1.8, we find that:

$(1.71)^2 = 2.9241$ $(1.75)^2 = 3.0625$

$(1.72)^2 = 2.9584$ etc.

11 $(1.73)^2 =$ ______ | 2.9929

12 $(1.74)^2 =$ ______ | 3.0276

If we look at the squares of 1.71, 1.72, etc., we

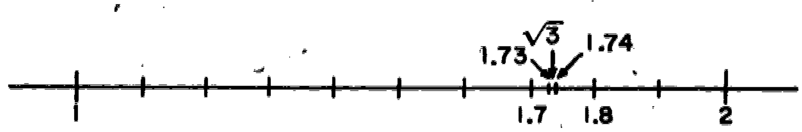
13 see that $(1.73)^2$ __ 3 and $(1.74)^2 > 3$. | $(1.73)^2 < 3$

14 Thus, we can write: ______ $< \sqrt{3} <$ ______. | 1.73, 1.74

To the nearest hundredth, the rational approximations

15 to $\sqrt{3}$ are ______ and ______. | 1.73, 1.74

16 $\sqrt{3}$ lies ______ 1.73 and 1.74. | between



Using the process we have just completed, let us find, to the nearest hundredth, rational approximations to $\sqrt{5}$.

Since $2^2 = 4$

17 $(\sqrt{5})^2 =$ ______, and | $(\sqrt{5})^2 = 5$

$3^2 = 9$,

18 ______ and ______ are the integers which are the closest approximations to $\sqrt{5}$. | 2, 3

19 We may write $2 \quad \sqrt{5} \quad 3$. — $2 < \sqrt{5} < 3$

Compute the following squares.

20 $(2.1)^2 =$ ______ — 4.41

21 $(2.2)^2 =$ ______ — 4.84

22 $(2.3)^2 =$ ______ — 5.29

23 ______ and ______ are rational approximations to — 2.2, 2.3

24 $\sqrt{5}$ to the nearest ______. — tenth

25 $(2.23)^2 =$ ______ — 4.9729

26 $(2.24)^2 =$ ______ — 5.0176

The approximations to $\sqrt{\ }$ to the nearest hundredth are 2.23 and 2.24.

To find an approximation to $\sqrt{7}$ we can locate $\sqrt{7}$

27 between the integers ______ and ______. — 2, 3

Find the approximations to $\sqrt{7}$ to the nearest

28 tenth. ______, ______ — 2.6, 2.7

The fact that $\sqrt{7}$ is between 2.6 and 2.7 can be written:

29 ___ $< \sqrt{7} <$ ___ — $2.6 < \sqrt{7} < 2.7$

We shall use the symbol " $\approx$ " to mean "is approximately equal to".

Since $\sqrt{7}$ is closer to 9 than it is to 4, we can say

30 $\sqrt{7}$ is approximately equal to ______. — 3

Using the symbol " $\approx$ ", we can write

31 $\sqrt{7}$ ______. — $\sqrt{7} \approx 3$

$\sqrt{5}$ is nearer to 2.2 than to 2.3, so we can write

32 $\sqrt{5}$ ______. — $\sqrt{5} \approx 2.2$

The process which we have been using to find rational approximations to $\sqrt{a}$ could be continued to thousandths, ten-thousandths, etc. Each time we would succeed in finding rational approximations, closer and closer together, with $\sqrt{a}$ lying between them. But this process is very slow and laborious. In the remainder of this section, we shall learn a more efficient method for finding an approximation to the square root of a number.

Our method for finding an approximation to the square root of a number will involve using the reasoning developed in the following items.

If $pq = 16$ and if $p = q$, where p and q are
33 positive, then $p = 4$ and $q =$ ______. — 4

34 $\sqrt{16} =$ ______. — 4

In general, if $pq = k$, and if $p = q$, then
35 $p = \sqrt{k}$ and $q =$ ______. — $\sqrt{k}$

If $pq = 16$ and if $p < 4$, then q must be
36 ________ than 4. — greater
(greater, less)

This must be true in order that $pq = 16$.

For example, if $pq = 16$ and p is 2, then
37 q is ______. Thus, q is greater than $\sqrt{16}$. — 8

In general, if $pq = k$ and if $0 < p < \sqrt{k}$, then
38 $q >$ _____. — $q > \sqrt{k}$

Let us now consider approximations to $\sqrt{10}$.

The successive integers between which $\sqrt{10}$ lies
39 are ______ and ______. — 3, 4

40 ___ $< \sqrt{10} <$ ___ — $3 < \sqrt{10} < 4$

41 $3^2 = 9$ and $4^2 =$ ______ — 16

10 is closer to 9 than to 16.

42 We conclude that ______ is the integer which is the best approximation to $\sqrt{10}$. — 3

We may indicate that $\sqrt{10}$ is approximately equal to 3
43 by writing $\sqrt{10}$ ______. — $\sqrt{10} \approx 3$

	For a second approximation to $\sqrt{10}$, consider the product $pq = 10$. (See Items 33-38.)	
	Let $p = 3$ (3 is a first approximation to $\sqrt{10}$).	
44	Then, for $pq = 10$ we may write: $3q =$ ______.	10
45	Since $3 < \sqrt{10}$, $q \underset{(<,=,>)}{\underline{\qquad}} \sqrt{10}$.	$>$
46	Since $3q = 10$, then $q =$ ______.	$\frac{10}{3}$, or $3\frac{1}{3}$
47	$\frac{10}{3}$ is approximately 3.33. Hence, $\sqrt{10}$ lies between ____ and ____.	3 and 3.33
	We take as a second approximation to $\sqrt{10}$ the point halfway between 3 and 3.33.	
	This is the average of 3 and 3.33, or $\frac{3 + 3.33}{2}$.	
48	$\frac{3 + 3.33}{2} = \frac{\square}{2}$	6.33
49	The second approximation to $\sqrt{10}$ is ______. (Round to 3 digits.)	3.17
50	$\sqrt{10} \approx$ ______.	3.17
	You can check to see just how close to $\sqrt{10}$ the second approximation is by squaring 3.17.	
51	$(3.17)^2 =$ ______	10.0489
	Thus, 3.17 is a good rational approximation to $\sqrt{10}$, and it was found with little effort.	

Refer to the following table as you complete Items 52-62.

Approximation to $\sqrt{22}$

Approximate	Divide	Average
p	$q = \frac{22}{p}$	$\frac{p + q}{2}$
5	$q = \frac{22}{5} = 4.40$	$\frac{5 + 4.40}{2} = 4.70$
4.7	$q = \frac{22}{4.7} \approx 4.681$	$\frac{4.7 + 4.681}{2} \approx 4.691$
$\sqrt{22} \approx 4.691$		

In finding an approximation to $\sqrt{22}$, we see that

52 $4^2 =$ ______ and $5^2 =$ ______. — 16, 25

The integer which is the best approximation to $\sqrt{22}$

53 is ______, since 5^2 is closer to 22 than is 4^2. — 5

54 If $pq = 22$ and if $p = 5$, then $q = \frac{22}{5} =$ ______. — 4.40

55 We are using ______ (what integer) as our first approximation. — 5

We get the second approximation to $\sqrt{22}$ by averaging the first approximation, p, which is 5, and q,

56 which is ______. — 4.40

$\frac{p + q}{2} = \frac{5 + 4.40}{2} = 4.70$, our second approximation.

57 $(4.70)^2 =$ ______. — 22.0900

To get a third and still closer approximation to $\sqrt{22}$, we round off the second approximation, 4.70, to two digits. We use this number, 4.7, as p.

We then find the corresponding q, where $pq = 22$.

58 $4.7q = 22$. $q = \frac{22}{4.7} \approx$ ______ (to the nearest thousandth). — 4.681

59 $\frac{p + q}{2} = \frac{\_\_\_ + 4.681}{2}$ — 4.7

60 $\approx$ ______ — 4.691

Hence, 4.691 is the third approximation to $\sqrt{22}$.

61 $(4.691)^2 =$ ______. — 22.005481

62 $\sqrt{22} \approx$ ______. — 4.691

Find the third approximation to $\sqrt{59}$.

63 Since $7^2 =$ ______ and $8^2 =$ ______, then a first — 49, 64

64 approximation for $\sqrt{59}$ is ______. — 8

$$\frac{59}{8} \approx 7.38$$

To find the second approximation to $\sqrt{59}$, we find

65 the ______ of 8 and 7.38. — average

66 $$\frac{8 + 7.38}{2} = \text{______}.$$ — 7.69

To get the third approximation, we first round off

67 7.69 to two digits (that is, to ______) and find, to the nearest thousandth — 7.7

68 $$\frac{59}{7.7} \approx \text{______}.$$ — 7.662

The average of 7.7 and 7.662 is

69 $$\frac{7.7 + 7.662}{2} = \text{______}.$$ — 7.681

Hence, 7.681 is the third approximation to $\sqrt{59}$.

70 $(7.681)^2 =$ ______ — 58.997761

71 $\sqrt{59} \approx$ ______ — 7.681

Find the third approximation to $\sqrt{19}$.

72 A first approximation to $\sqrt{19}$ is ______. — 4

73 $\frac{19}{4} =$ ______ (to 3 digits) — 4.75

$$\frac{4 + 4.75}{2} = 4.375.$$

74 The second approximation to $\sqrt{19}$ is ______. — 4.38

75 $\frac{19}{4.4} \approx$ ______ (to 4 digits) — 4.318

76 $\frac{4.4 + 4.318}{2} =$ ______ — 4.359

The third approximation to $\sqrt{19}$ is 4.359.

77 $(4.359)^2 =$ ______ — 19.000881

$\sqrt{19} \approx 4.359.$

Find the third approximation for each of the following:

78 $\sqrt{42} \approx$ ______ — 6.481

79 $\sqrt{74} \approx$ ______ — 8.602

80 $\sqrt{93} \approx$ ______ — 9.644

If you had trouble with any of these, turn to page xii. and check your work.

15-6. Approximate Square Roots of Positive Numbers

In Section 15-5 we have seen how to find an approximation to $\sqrt{a}$ where a is a number between 1 and 100. To recognize the first approximation, we needed only to recall the squares of the positive numbers less than or equal to 10.

1	Since $8^2 = 64$ and $9^2 = 81$, $\sqrt{71}$ is between _____ and _____.	8 and 9
2	Of the two numbers 64 and 81, 71 is nearer to _____.	64
3	Hence, a first approximation to $\sqrt{71}$ is _____.	8
4	$\sqrt{30}$ is between _____ and _____.	5 and 6
5	A first approximation to $\sqrt{30}$ is _____.	5
6	$\sqrt{2.8}$ is between _____ and _____.	1 and 2
7	A first approximation to $\sqrt{2.8}$ is _____.	2

If we wish closer approximations to $\sqrt{71}$, $\sqrt{30}$, and $\sqrt{2.8}$, we can "divide and average" as we did in Section 15-5.

Suppose we wish to find approximations to square roots of positive numbers which are <u>not</u> between 1 and 100.

Although we <u>could</u> use exactly the same process as before, the work of finding $\sqrt{x}$ will often be easier if we write x as the product of two numbers of which one is between 1 and 100, and the other is an even power of 10.

8	$10^2 =$ _____	100
	$356 = 3.56 \times 100$	
9	$= 3.56 \times 10^{\square}$	3.56×10^2
	$\sqrt{356} = \sqrt{3.56 \times 10^2}$	
	$= \sqrt{3.56} \cdot \sqrt{10^2}$	
10	$= \sqrt{}(10)$	$\sqrt{3.56}(10)$
	If the third approximation to $\sqrt{3.56}$ is 1.887,	
	then $\sqrt{356} \approx 1.887(10)$	
11	or $\sqrt{356} \approx$ _____	18.87
12	$(18.87)^2 =$ _____	356.0769

Notice that we have made use of Theorem 15-3, "For any positive numbers a and b, $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$" to show that $\sqrt{356} = \sqrt{3.56} \cdot \sqrt{10^2}$.

The general method used could be summarized as follows:

To find an approximation to $\sqrt{x}$, for x any positive number,

1) rename x as $a(10^{2n})$ where $1 < a < 100$ and n is an integer.

2) Then $\sqrt{x} = \sqrt{a(10^{2n})} = \sqrt{a}(10^n)$

3) Use the divide and average method of Section 15-5 to get an approximation for $\sqrt{a}$.

4) Multiply this result by 10^n to have an approximation for $\sqrt{x}$.

Let us look at some even powers of 10.

13	$\underline{10^{\square}} = 100$	10^2
14	$10^4 =$ ______	10,000
15	$10^{-2} = \dfrac{1}{10^{\square}}$	$\dfrac{1}{10^2}$
16	$=$ ______ $= 0.01$	$\dfrac{1}{100}$
17	$0.0001 = \dfrac{1}{10^4} = 10^{\square}$	10^{-4}
18	$10^0 =$ ______	1

Now we might try writing some positive numbers in the form $a(10^{2n})$, where $1 < a < 100$ and n is an integer.

19	$392 \times 10^{-2} =$ ______	3.92
20	Hence, $392 = \underline{3.92(10^{\square})}$	$3.92(10^2)$
21	$0.392 \times 10^2 =$ ______	39.2
22	Hence, $0.392 =$ ______(10^{-2})	$39.2(10^{-2})$
23	$8752 \times \underline{10^{\square}} = 87.52$	10^{-2}
24	Hence, $8752 = \underline{87.52(\quad)}$	$87.52(10^2)$
25	$0.08 \times 10^2 =$ ______	8
26	Hence, $0.08 = \underline{8(\quad)}$	$8(10^{-2})$
27	$0.8 \times 10^2 =$ ______	80
28	Hence, $0.8 =$ ______(10^{-2})	$80(10^{-2})$

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29	$0.009956 \times 10^4 =$ ______	99.56
30	Hence, $0.009956 =$ ______ (10^{-4})	$99.56(10^{-4})$
31	$93{,}000{,}000 \times 10^{\square} = 93$	10^{-6}
32	Hence, $93{,}000{,}000 = 93(\quad)$	$93(10^6)$
33	$35{,}000 \times 10^{\square} = 3.5$	10^{-4}
34	Hence, $35{,}000 =$ ______	$3.5(10^4)$

	Write each of the following numbers in the form $a(10^{2n})$, where $1 < a < 100$ and n is an integer.	
35	$0.0294 =$ ____ $(10^{\square})$	$2.94(10^{-2})$
36	$0.7691 =$ ______	$76.91(10^{-2})$
37	$173 =$ ______	$1.73(10^2)$
38	$600 =$ ______	$6.00(10^2)$
39	$8168 =$ ______	$81.68(10^2)$
40	$128.67 =$ ______	$1.2867(10^2)$
41	$0.00392 =$ ______	$39.2(10^{-4})$

Notice that in each case, the number could have been renamed in another way, which would also be a product of two numbers such that one is a number between 1 and 100 and the other is some power of 10.

42	For 8168, we wrote $81.68(\quad)$;	$81.68(10^2)$
43	we could also write $8168 = 8.168(10^{\square})$.	$8.168(10^3)$
44	For 0.0294, we wrote $2.94(\quad)$;	$2.94(10^{-2})$
45	we could also write $0.0294 = 29.4(10^{\square})$.	$29.4(10^{-3})$
46	For 173, we wrote $1.73(\quad)$;	$1.73(10^2)$
47	we could also write $173 = 17.3(\quad)$	$17.3(10^1)$
48	Note that in each case our first choice was that in which the power of 10 is ______ (even, odd).	even

49	Which of the following are rational numbers? $\sqrt{10^2}$, $\sqrt{10^{-4}}$, $\sqrt{10^{-3}}$, $\sqrt{10^5}$, $\sqrt{10^4}$ [A] $\sqrt{10^2}$, $\sqrt{10^5}$, $\sqrt{10^4}$ [B] $\sqrt{10^{-4}}$, $\sqrt{10^{-3}}$ [C] $\sqrt{10^2}$, $\sqrt{10^{-4}}$, $\sqrt{10^4}$	

$\sqrt{10^2} = 10$, and 10 is rational. Also, $\sqrt{10^{-4}} = \frac{1}{100}$ and $\sqrt{10^4} = 100$. Both $\frac{1}{100}$ and 100 are rational. However, $\sqrt{10^{-3}} = \frac{1}{100} \cdot \sqrt{10}$ and $\sqrt{10^5} = 100\sqrt{10}$. Both of these numbers are irrational, since $\sqrt{10}$ is irrational. Hence, [C] is the correct choice.

As illustrated above, the square root of a power of 10 is a rational number only if the exponent is even. This is why, when we wish to find an approximation to $\sqrt{x}$, we first rename x by a numeral of the form $a(10^{2n})$ where $1 < a < 100$ and n is an integer; that is, where 10^{2n} is an <u>even</u> power of 10.

50	Since $2730 = \underline{27.3(10^{\square})}$, we have $\sqrt{2730} = \sqrt{27.3} \cdot \sqrt{10^2}$	$27.3(10^2)$
51	$= \underline{\sqrt{}(10)}$	$\sqrt{27.3}(10)$
	Since $\sqrt{27.3}$ is between 5 and 6, but is closer to 5, we see that $\sqrt{2730}$ is between	
52	_______ and _______, and is closer to 50.	50 and 60
53	Thus, a first approximation to $\sqrt{2730}$ is _______.	50

If we wish to find a closer approximation to $\sqrt{2730}$, we can approximate $\sqrt{27.3}$ by the "divide and average" process, and then multiply the resulting approximation by 10.

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A first approximation to $\sqrt{27.3}$, as we saw above,

54 is ______. | 5

55 Divide: $\frac{27.3}{5}$ = ______ | 5.46

56 Average: $\frac{5 + 5.46}{2}$ = ______ | 5.23

The second approximation to $\sqrt{27.3}$ is 5.23.

57 Divide: $\frac{27.3}{5.2}$ = ______ | 5.250

58 Average: $\frac{5.2 + 5.250}{2}$ = ______ | 5.225

The third approximation to $\sqrt{27.3}$ is 5.225.

Thus, the third approximation to $\sqrt{2730}$ is

59 (5.225)(10), or ______. | 52.25

60 $(52.25)^2$ = ______ | 2730.0625

Find the second approximation to $\sqrt{354,000}$.

61 Since 354,000 = ______ (10^4), | $35.4(10^4)$

$\sqrt{354,000} = \sqrt{35.4}\sqrt{10^4}$

62 $= \sqrt{35.4}(10^{\square})$ | $\sqrt{35.4}(10^2)$

35.4 is very close to 36, so a good first approx-

63 imation to $\sqrt{35.4}$ is ______. | 6

64 Divide: $\frac{35.4}{6}$ = ______ | 5.90

65 Average: $\frac{6 + 5.90}{2}$ = ______ | 5.95

The second approximation to $\sqrt{35.4}$ is 5.95, so the second approximation to $\sqrt{354,000}$ is $(5.95)(10^2)$,

66 or ______. | 595

67 $(595)^2$ = ______ | 354,025

Find the third approximation to $\sqrt{0.2138}$.

68 0.2138 = $21.38(10^{\square})$ | $21.38(10^{-2})$

Since 21.38 is between 16 and 25, but is

69 closer to ______, a good first approximation to | 25

70 $\sqrt{21.38}$ is ______. | 5

71 Divide: $\frac{21.38}{5} \approx$ ______ 4.28

72 Average: $\frac{5 + 4.28}{2} =$ ______ 4.64

The second approximation to $\sqrt{21.38}$ is 4.64.

73 Divide: $\frac{21.38}{4.6} \approx$ ______ 4.648

74 Average: $\frac{4.6 + 4.648}{2} =$ ______ 4.624

The third approximation to $\sqrt{21.38}$ is 4.624, so the

75 third approximation to $\sqrt{0.2138}$ is $4.624(10^{\square})$. $4.624(10^{-1})$

76 $4.624(10^{-1}) =$ ______ 0.4624

77 $(0.4624)^2 =$ ______ [illegible]

Find the third approximation to each of the following:

78 $\sqrt{0.00470} \approx$ ______ [illegible]

79 $\sqrt{0.0470} \approx$ ______ [illegible]

80 $\sqrt{70260} \approx$ ______ [illegible]

If you had trouble with any of these, check your work with that shown on page xiii.

Find the third approximation to

81. $\sqrt{1681}$. ______

82. $\sqrt{0.1369}$. ______

Check your work with that on page xiv.

Find the second approximations to the elements of the truth set of

$$x^2 = 1.24.$$

83 The truth set is ______. $\{\sqrt{1.24}, -\sqrt{1.24}\}$

84 The approximations are ______ and ______. [illegible]

The usefulness of a printed table of decimal approximat[illegible] roots of numbers is increased if we use what we have learne[illegible] a number in a form which involves an even power of 10.

For example: A table of square roots gives
$\sqrt{72} \approx 8.485$.

35	$\sqrt{720{,}000} = \sqrt{\_\_\_\_} \cdot \sqrt{10^4}$	$\sqrt{72}$
36	$= 8.485(10^2)$, or ____	848.5
37	$\sqrt{0.0072} = \sqrt{72} \cdot \sqrt{10^{-4}} \approx$ ____	0.08485
38	$\sqrt{0.72} = \sqrt{72} \cdot \sqrt{\_\_\_\_} \approx$ ____	$\sqrt{10^{-2}}$, 0.8485
39	$\sqrt{7200} = \sqrt{72} \cdot \sqrt{\_\_\_\_} \approx$ ____	$\sqrt{10^{2}}$, 84.85

If $\sqrt{8} \approx 2.828$, then

40	$\sqrt{0.08} \approx$ ____	0.2828
41	$\sqrt{80{,}000} \approx$ ____	282.8
42	$\sqrt{800} \approx$ ____	28.28
43	$\sqrt{0.0008} \approx$ ____	0.02828

	Does the information that $\sqrt{8} \approx 2.828$ help you to	
44	find $\sqrt{80}$? ____ (yes, no)	no
45	Since $\sqrt{80} = \sqrt{8} \cdot \sqrt{\_\_\_\_}$, and $\sqrt{10}$ is not rational,	$\sqrt{8} \cdot \sqrt{10}$
	$\sqrt{80}$ does not equal the product of $\sqrt{8}$ and an	
46	integral power of ____.	10

15-7. Summary and Review

If b is a positive real number, then there is exactly one positive number whose square is b. We define:

> If b is a positive real number, then $\sqrt{b}$ is the positive number whose square is b.

The negative number whose square is b is denoted by $-\sqrt{b}$.

We also defined: $\sqrt{0} = 0$.

If x is any real number,

$$\sqrt{x^2} = |x|$$

That is:

$$\sqrt{x^2} = x \quad \text{if} \quad x \geq 0$$
$$\sqrt{x^2} = -x \text{ if } \quad x < 0.$$

We also defined: If b is a real number, and $a^3 = b$, then $a = \sqrt[3]{b}$.

We have proved:

Theorem 15-2. $\sqrt{2}$ is irrational.

Theorem 15-3. $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ for any non-negative numbers a and b.

Theorem 15-4. If $a \geq 0$ and $b > 0$, then $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.

To approximate $\sqrt{x}$, where x is positive, we have used the following method.

First, write x in the form $a(10^{2n})$ where $1 < a < 100$ and n is an integer. Hence,

$$\sqrt{x} = \sqrt{a}\sqrt{10^{2n}}$$

Second, find an integer p between 1 and 10 inclusive which is the first approximation to $\sqrt{a}$. To find the second approximation, divide a by p to find q, $(q = \frac{a}{p})$, and determine the average of p and q. This average, $\frac{p + q}{2}$, rounded off to two digits, is the second approximation to $\sqrt{a}$.

Then $$\sqrt{x} \approx \frac{p + q}{2} \times 10^n.$$

If more accuracy is desired, use the second approximation of $\sqrt{a}$ as the new value of p and carry out the division $\frac{a}{p}$ to four digits. Use these new values of p and q to find the average $\frac{p + q}{2}$.

Review Problems

Answers to the review problems are on page xv.

1. Simplify, indicating the domain of the variable when it is restricted.

(a) $\sqrt{12}$ (d) $\sqrt{\frac{16}{3}}$ (rationalize the denominator) (g) $\sqrt{8} - \sqrt{18}$

(b) $\frac{1}{\sqrt{36}}$ (e) $\sqrt{18x^2}$ (h) $(a^2bc)(ab^2c)$

(c) $\sqrt{8a}$ (f) $\sqrt{6}\ \sqrt{24}$ (i) $\sqrt{2}(\sqrt{2} + \sqrt{6})$

2. Simplify, indicating the domain of the variable when it is restricted.

(a) $\sqrt{48} - \sqrt{75} + \sqrt{12}$

(b) $\sqrt{5}\ \sqrt{20}$

(c) $\sqrt{4(a + b)^2}$

(d) $\sqrt{2}(\sqrt{2} - \sqrt{6})$

(e) $\sqrt{\frac{1}{2}} - \frac{\sqrt{9}}{\sqrt{8}}$

(f) $\frac{\sqrt{3}}{\sqrt{2}} + 1$

(g) $\sqrt[3]{54}$

3. Rationalize the denominator. Indicate the domain of the variable in each case where it is restricted.

(a) $\sqrt{\frac{1}{12}}$

(b) $\sqrt{\frac{5}{18}}$

(c) $3\sqrt{\frac{7}{36}}$

(d) $\frac{\sqrt{240}}{\sqrt{24}}$

(e) $\frac{9\sqrt{5}}{12\sqrt{15}}$

(f) $\sqrt{\frac{2x^2}{9}}$

(g) $\sqrt{\frac{5}{x^3}}$

*(h) $\sqrt[3]{\frac{3}{4}}$

*(i) $\sqrt[3]{\frac{1}{9a^2}}$

4. Simplify, indicating the domain of the variable when it is restricted, and rationalizing denominators.

(a) $2\sqrt{12a^2} - \frac{3|a|}{\sqrt{3}} - \frac{1}{4}\sqrt{48a^2}$

(b) $\frac{\sqrt{3} + \sqrt{2}}{\sqrt{2}}$

(c) $(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})$

(d) $\sqrt{\frac{4m^2}{q}} + \sqrt{98m^2q^3}$

(e) $\sqrt{\frac{2}{3}}\ \sqrt{\frac{5}{6}}\ \sqrt{\frac{3}{2}}$

(f) $\sqrt{3p}\ \sqrt{6p^3}$

(g) $\sqrt{4a^2 + 4b^2}$

*(h) $\frac{1}{\sqrt[3]{250x}}$

*(i) $\sqrt[3]{\frac{1}{4a}} - \sqrt[3]{16a^5}$

5. Solve the following:

(a) $\sqrt{x} = 2$

*(b) $\sqrt[3]{n} = 4$

(c) $y^2 = 2$

(d) $m^2 \leq 16$

(e) $3 = \sqrt{t + 1}$

$2|x| + \sqrt{x^2} = 3$

6. In each of the following use one of the symbols $<$, $=$, $>$ between the two given phrases so as to make a true sentence.

   (a) $\frac{1}{x} + \frac{2}{3x}$, $\frac{1}{3}$, for x [illegible] 5 (b) $x + \sqrt{2}$, $\sqrt{2}$, for $x > 0$

7. Evaluate $\sqrt{3400}$ (to the third approximation).

8. $(62.4[illegible])^2$ = ______

9. Express as powers of [illegible] numerals, if possible.

   (a) [illegible] (d) $[illegible]^2 + [illegible]^2 + [illegible]$

   (b) [illegible] (e) [illegible]

   (c) $3^2 \cdot 2^3$ (f) $3^2 + 2^2$

10. Express as powers of 10. (n is an integer.)

   (a) $10^{-2} \times 10^2$ (c) $10^n \times 10^2$ (e) $10^{-3} \times 10^{-5} \times 10^3$

   (b) $\frac{10^3 \times 10^{-1}}{10^1}$ (d) $\frac{10^2 \times 10^{-3}}{10^{-2}}$ (f) $(10^{2n})^3$

11. Simplify: (Assume no variable takes on the value zero.)

   (a) $\frac{\frac{2}{3}}{\frac{5}{6}}$ (d) $\frac{\frac{1}{4} + \frac{1}{3}}{\frac{5}{12}}$

   (b) $\frac{\frac{1^3}{2}}{\frac{b}{6}}$ (e) $\frac{1 + \frac{1}{x}}{1 - \frac{1}{x}}$

   (c) $\frac{\frac{mn}{q^2}}{\frac{2nq}{m^2}}$ (f) $\frac{\frac{x + 3}{2}}{\frac{x(x + 3)}{3}}$

12. Solve:

   (a) $\frac{1}{3}x - 1 > \frac{1}{2}x$ (c) $\frac{y}{3} + \frac{y}{5} < 1$

   (b) $\frac{3 - x}{3}$ [illegible] $\frac{10x}{6}$ (d) $|m| = \frac{3}{20} - \frac{1}{5}|m|$

13. A remarkable expression which produces many primes is

$$n^2 - n + 41.$$

If n is any number of the set $\{1,2,3, \ldots, 40\}$ the value of the expression is a prime number, but for $n = 41$ the expression fails to give a prime number. Tell why it fails. If an algebraic sentence is true for the first 400 values of the variable, is it then necessarily true for the 401st?

14. A procedure sometimes used to save time in averaging large numbers is to guess at an average, average the differences, and add that average to your guess. Thus, if the numbers to be averaged--say your test scores--are 78, 80, 76, 72, 85, 70, 90, a reasonable guess for your average might be 80. We find how far each of our numbers is from 80.

$78 - 80 = -2$	
$80 - 80 = 0$	The sum of the differences is -9.
$76 - 80 = -4$	The average of the differences is
$72 - 80 = -8$	$-\frac{9}{7}$. Adding this to 80 gives
$85 - 80 = 5$	$78\frac{5}{7}$ for the desired average. Can
$70 - 80 = -10$	you explain why this works?
$90 - 80 = 10$	

The weights of a university football team were posted as 195, 205, 212, 201, 198, 232, 189, 178, 196, 204, 182. Find the average weight for the team by the above method.

15. A rat which weighs x grams is fed a rich diet and gains 25% in weight. He is then put on a poor diet and loses 25% of his weight. Find the number of grams difference in the weight of the rat from the beginning of the experiment to the end.

Chapter 16

POLYNOMIALS AND FACTORING

16-1. Polynomials

In Chapter 12, we studied extensively the integers and the factorization of integers. We were particularly interested in expressing positive integers in terms of their prime factors. We saw also that factoring helped us in working with fractions and radicals.

Since the factored form for integers has turned out to be so useful, it is natural to ask whether we can write algebraic phrases in factored form, that is, as indicated products of simpler phrases.

We have often written such sentences as

1 $$3x^2 + 2x = x(3x + \_\_\_\_).$$ (3x + 2)

2 We recognize that this sentence, which is true for all values of x, illustrates the ______ property. distributive

3 We have written $3x^2 + 2x$ as a product of two ______. factors

We also know that there are many other ways of writing $3x^2 + 2x$ as a product. For example,

4 $$3x^2 + 2x = \_\_\_\_\_\_ \cdot \frac{3x^2 + 2x}{x^2 + 1}$$ $x^2 + 1$

This is not, however, a simpler form.

The expression $3x^2 + 2x$ involves only the operations of addition and multiplication, but the factor $\frac{3x^2 + 2x}{x^2 + 1}$ involves addition, multiplication, and

5 ______. division

We begin to suspect that in factoring algebraic expressions we should restrict ourselves to certain kinds of factors.

Examine the phrases:

P. $\frac{x^2 + 5}{2x - 1}$ R. $5x^2 + 3x - \sqrt{7}$

Q. $x^2 + 3x$ S. $\frac{4}{x + 3}$

Each of these involves only real numbers and the single variable x. If we were asked how Q and R were different from the others, we would say that neither phrase involves the operation of:

[A] addition [C] subtraction

[B] multiplication [D] division

Q clearly involves addition and R clearly involves subtraction. Since $x^2 = x \cdot x$ and $3x = 3 \cdot x$, each phrase also involves multiplication. [D] is the correct choice.

The phrases $x^2 + 3x$ and $5x^2 + 3x - \sqrt{7}$ are examples of polynomials. These phrases do not involve any indicated division by a phrase containing a variable.

It will turn out, as you will see, that polynomials play, in connection with algebraic expressions, a role similar to that of the integers in the set of real numbers.

Consider a set consisting of the real numbers and one or more variables. Any phrase formed from members of this set by using no indicated operations other than addition, subtraction, or multiplication is called a polynomial.

According to our definition, 5 is a polynomial. In the same way 107, -15, x, y, 0, $\frac{3}{4}$ are polynomials.

Other examples of polynomials are:

$$x - \sqrt{5},\quad 5x,\quad x^5,\quad (5 - x)^{10},\quad x^2 - 2x - 3,\quad \text{and}\quad \frac{4}{3}x.$$

Which of the following are polynomials?

P. x^3 R. $x(x + 3)$

Q. $\frac{x^2 - 1}{x}$ S. $(x + 1) + (2x + 5)$

T. $\sqrt{x} - .06$

[A] P, Q, R, S

[B] all

[C] P, R, S

[D] P, R, S, T

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Q involves division, and T involves the extraction of a square root of a variable. P, R, and S are polynomials. [C] is the correct choice.

	For each of the following phrases, indicate whether it is or is not a polynomial:		
8	$3t + 1$	______ (is, is not)	is
9	$t + \frac{1}{3}$	______	is
10	$3a^2b$	______	is
11	$at - \sqrt{2}$	______	is
12	$\|x\| + 1$	______	is not [involves $\|x\|$]
13	$(s + 5)(t - 1)v$	______	is
14	$\frac{2}{3}(x - 4)$	______	is
15	$\frac{x + 1}{x - 1}$	______	is not [involves division]
16	$-\frac{1}{2}$	______	is

If only one variable appears in a polynomial, we have a polynomial in that variable. For example, if the variable is x, we have a <u>polynomial in x</u>.

	$3x - 1$ is a polynomial in x.	
17	Likewise $4y^2 - y$ is a polynomial in ____.	y
18	$4y + 3x$ is a polynomial in ____ and ____.	x, y

	$x^2 - 5x + 6$ is a polynomial.	
	$x^2 - 5x + 6$ may be thought of as the <u>sum</u> of three terms: x^2, $-5x$, and 6.	
19	Each of these terms is also a ______.	polynomial.

An expression such as $-5x$ has only one term. We call such an expression a <u>monomial</u>. In general, a polynomial which involves at most indicated products is called a monomial.

20	$-6x^3y$ ______ (is, is not) a monomial.	is
21	$x + 6y$ ______ (is, is not) a monomial.	is not
22	$-6x^3y$ ______ (is, is not) a polynomial.	is
23	$x + 6y$ ______ (is, is not) a polynomial.	is

We have had experience in simplifying indicated sums by "collecting terms". For example,

$$(x + 2) + (x + 3) = 2x + 5.$$

Similarly,

24	$(x^2 - x + 2) + (3x - 1) =$ ______	$x^2 + 2x + 1$
25	$(x^3 + 2x^2 - x + 3) - (x^3 - x^2) =$ ______	$3x^2 - x + 3$
26	$(1 - x) + (1 + x) =$ ______.	2
27	$(xy + y^2 - x) + (x^2 + y^2 + 2x) = x^2 + xy +$ ______ $+$ ______	$x^2 + xy + 2y^2 + x$

We can use the distributive property to write:

28	$4x(1 - x) = 4x$ ______	$4x - 4x^2$
29	$2(x + 1) + 3(2x - 3) = 5x$ ______	$8x - 7$
30	$(x - 2)(x - 3) = x(x - 3) - 2(x - 3) =$ ______	$x^2 - 5x + 6$

These examples, together with our previous experience, tell us that we can write any polynomial as a sum of monomials. Furthermore, the process of collecting terms can be applied to every sum of monomials until further simplification by this process is impossible.

When a polynomial has been written as a sum of monomials and the process of collecting terms has been completed we say that the polynomial has been written in <u>common polynomial form</u>. When writing polynomials in one variable in common polynomial form it is convenient to arrange the terms so that the powers of the variable are in descending order.

31	$7y - 8 + 10y^4 + y^2$ is in common polynomial form. If we write this polynomial with descending powers of y, we have ______.	$10y^4 + y^2 + 7y - 8$

By the <u>degree</u> <u>of</u> <u>a</u> <u>polynomial</u> in one variable, we mean the highest power of the variable that occurs when the polynomial is written in common polynomial form. (This is one reason it is convenient to arrange the terms so that the powers of the variable are in descending order.)

32	$3x^3 - 2x + 4$ is a polynomial of ______ 3.	degree
33	$1 + x^2 - 5x$ is a polynomial of <u>degree</u> ______.	degree 2
	State the degrees of the following polynomials:	
34	$\frac{1}{2} - x^3 + 5$ ______	3
35	x^{17} ______	17
36	$x^3 + x^2 - x - x^3$ ______ [careful!]	2
37	$3(x - 1) + x(2x + 5)$ ______ [First write in common polynomial form.]	2
38	7 ______ [$7 = 7 \cdot 1 = 7x^0$]	0

Notice that polynomials such as 3, -1, 8, etc., have degree 0 (see Item 38). However, for technical reasons which you will learn about in later courses we do <u>not</u> define the degree of the polynomial 0. Thus every polynomial <u>except</u> <u>0</u> has a degree. (By now you should be familiar with the fact that a situation involving 0 often presents exceptional features.)

We shall often be concerned with polynomials of degree 2. Such polynomials are called <u>quadratic</u> <u>polynomials</u>.

39	$3x^2 - 2x + 4$ is a quadratic ______.	polynomial
40	$5 - x$ ______ (is, is not) a quadratic polynomial.	is not
41	$2x - x^2 + 5$ is a ______ polynomial.	quadratic

Now let us consider sums, products, and differences of polynomials.

$x + 2$ and $x^2 + x - 5$ are polynomials in common polynomial form.

Notice that: $-(x + 2)$

$(x + 2) + (x^2 + x - 5)$

$(x + 2) - (x^2 + x - 5)$

and $(x + 2)(x^2 + x - 5)$

are also polynomials.

Write each of the four polynomials listed in common polynomial form:

42 $-(x + 2) =$ ______ $-x - 2$

43 $(x + 2) + (x^2 + x - 5) =$ ______ $x^2 + 2x - 3$

44 $(x + 2) - (x^2 + x - 5) =$ ______ $-x^2 + 7$

45 $(x + 2)(x^2 + x - 5) = x(x^2 + x - 5) + 2(x^2 + x - 5) =$ ______ $x^3 + 3x^2 - 3x - 10$

By the instruction, "add two polynomials", we shall mean, "write the indicated sum in common polynomial form", as in Item 43 above. Similarly, the instructions for Item 44 might be: "subtract the polynomial $x^2 + x - 5$ from the polynomial $x + 2$." In Item 45 we might say, "multiply the polynomials."

46 Add: $y^2 - 3$ and $-2y^2 + 4y - 8$. $-y^2 + 4y - 11$

47 Multiply: $t^2 + 3$ by $t^2 - 2$. $t^4 + t^2 - 6$

48 From $8 + x^2 - 3x + x^3$ subtract $x - 3 + 2x^2$. $x^3 - x^2 - 4x + 11$

In organizing your work in Item 48 you have perhaps discovered another advantage of arranging the terms of a polynomial in order of descending powers of the variable. The work might be done as follows:

$$\begin{array}{lr} & x^3 + x^2 - 3x + 8 \\ \text{subtract} & \underline{\quad 2x^2 + \ x - 3} \\ & x^3 - x^2 - 4x + 11 \end{array}$$

49 The opposite of the polynomial $5 - x$ is ______. $x - 5$

50 Add $2x^2 + x - 2$ and $2 - x - 2x^2$. ______ 0

51 0 ______ (is, is not) a polynomial. is

We can observe from the examples above that the set of polynomials is closed under addition, subtraction, and multiplication. This observation points up a similarity between the set of polynomials and the set of integers. The set of integers is closed under the same operations. Notice, moreover, that neither the set of integers nor the set of polynomials is closed under division.

[illegible]

$$3x^2 - 2x + 4$$

it is [illegible] to have a means of identifying the integers 3, -2, and 4.

In $3x^2 - 2x + 4$

the coefficient of x^2 is 3,

the coefficient of x is -2 and

the "constant" is 4.

For convenience, we may speak of the set of coefficients of $3x^2 - 2x + 4$ as the set whose elements are 3, -2, and 4.

	Given $2x - x^2 + 5$,	
52	the coefficient of x^2 is ______,	-1
53	the coefficient of x is ______,	2
54	the constant is ______,	5
55	the set of coefficients is ______.	{-1,2,5}
56	In $4x^3 - 3x + 2$, the coefficient of x^2 is ______, since we could write $4x^3 + 0x^2 - 3x + 2$.	0
57	In $4 - 3x^2 + 5x$ the constant is ______, and	4
58	the coefficient of the term of highest degree is ______.	-3
59	The coefficients of $\sqrt{2}x^3 - 4x$, in descending powers of x, are _____, _____, _____, and _____.	$\sqrt{2}$, 0, -4, 0

60 Which of the following is a quadratic polynomial that has 0 for its constant?

[A] $x^2 - x^3$ [C] $x^2 - \frac{1}{x}$

[B] $x^2 + x$ [D] $x^2 - 1$

[A] is of degree 3. [C] is not a polynomial at all. [D] has -1 for its constant. [B] is the only correct choice.

61	In the polynomial $4x^3 - 8x^2 + 2x - 3$ the set of coefficients is ______.	$\{4, -8, 2, -3\}$
	The numbers $4, -8, 2, -3$ are all <u>integers</u>.	
	We may call $4x^3 - 8x^2 + 2x - 3$ a <u>polynomial over the integers</u>.	
62	$5x + 6$ is also a polynomial over the ______, since 5 and 6 are integers.	integers
63	In $7x^2 + \frac{4}{3}x + 5$, the coefficient ____ is not an	$\frac{4}{3}$
64	integer. Hence $7x^2 + \frac{4}{3}x + 5$ ______ (is, is not) a polynomial over the integers.	is not
65	The constant in $x - \frac{1}{4}$ ______ (is, is not) an integer, and	is not
66	hence $x - \frac{1}{4}$ is not a polynomial over the ______.	integers
67	$x - \sqrt{2}$ ______ (is, is not) a polynomial over the integers.	is not

Now we are ready to consider factoring polynomials. At the beginning of this section we considered the phrase $3x^2 + 2x$. It is a polynomial over the integers.

It is easy to write $3x^2 + 2x$ as an indicated product.

68	$3x^2 + 2x = x(\quad\quad)$	$x(3x + 2)$
69	$= \frac{1}{x}(3x^3 + \quad)$	$\frac{1}{x}(3x^3 + 2x^2)$
70	$= \frac{1}{2}x(\quad\quad)$	$\frac{1}{2}x(6x + 4)$
71	$= \quad(x^2 + \frac{2}{3}x)$	$3(x^2 + \frac{2}{3}x)$

Of the four factorizations in Items 68 to 71, that of 68--namely, $x(3x + 2)$--is the simplest. What can we say of the others?

72 In Item 69 we saw: $3x^2 + 2x = \frac{1}{x}(3x^3 + 2x^2)$. In this less simple product, the factor $\frac{1}{x}$ $\underline{\hspace{3cm}}$ (is, is not) a polynomial. | is not

73 In Item 70 we saw: $3x^2 + 2x = \frac{1}{2}x(6x + 4)$. $\frac{1}{2}x$ and $6x$ $\underline{\hspace{3cm}}$ (are, are not) both polynomials. | are

74 However $\frac{1}{2}x$ $\underline{\hspace{3cm}}$ (is, is not) a polynomial over the integers. | is not

75 Likewise in Item 71 the factor $x^2 + \frac{2}{3}x$ is not a polynomial over the _______. | integers

Of the products in Items 69 to 71, the simplest is $x(3x + 2)$. In this product the factors x and $3x + 2$ are both polynomials over the integers.

When we are working with polynomials over the integers, we are often interested only in those factorizations in which all the factors are polynomials over the integers.

If we restrict ourselves to polynomials over the integers, then the factorizations of $12x^2 + 36x$ are:

76 $12x(\underline{\hspace{2cm}})$ | $12x(x + 3)$

77 $\underline{\hspace{0.5cm}}(x^2 + 3x)$ | $12(x^2 + 3x)$

78 $6x(\underline{\hspace{2cm}})$ | $6x(2x + 6)$

Of the expressions: $12x(x + 3)$

$12(x^2 + 3x)$

$6x(2x + 6)$

we prefer the first, $12x(x + 3)$. This is true because neither of the factors containing a variable can be factored further.

79 In factoring $12x^2 + 36x$, we do not, as a rule, stop with $12(x^2 + 3x)$, since $x^2 + 3x$ $\underline{\hspace{3cm}}$ (can, cannot) be factored further. | can

However, we usually write $12x(x + 3)$ as the final form, without factoring 12.

80	It is true that $x + 3 = (1)(\_\_\_\_)$. However, just as in Chapter 12, we do not use 1 as a factor in the final form.	$x + 3$

Factors such as x and $x + 3$ cannot be factored further. We will say that we have factored a polynomial <u>completely</u> when no factor containing a variable can be factored further. Polynomials which cannot be factored further play, in this chapter, a role similar to that of prime numbers in factoring integers.

We shall be working in much of this chapter with the problem of writing polynomials over the integers as products of polynomials over the integers. We shall speak of doing this as <u>factoring polynomials over the integers</u>.

81	As you already know, the ______ property can be used to write indicated sums as indicated products. Factor each of the following polynomials completely over the integers.	distributive
82	$3t - 6 = \_\_\_\_\_\_$	$3(t - 2)$
83	$ab + ac = \_\_\_\_\_\_$	$a(b + c)$
84	$ax - ay = \_\_\_\_\_\_$	$a(x - y)$
85	$2x^2 + 6x = \_\_\_\_\_\_$	$2x(x + 3)$

16-2. <u>Factoring by the Distributive Property</u>

In the preceding section we saw a few examples in which the distributive property was used to change an indicated sum to an indicated product. By now you should be very familiar with the distributive property. However, you will find that it takes practice to apply it in complicated situations.

	The pattern is easy to see in:	
	$3x + 3y$	
1	$3x + 3y = (\_\_\_\_\_)$	$3(x + y)$
2	$3x + 3y$ is the indicated sum of $3x$ and ____.	$3y$
3	Both $3x$ and $3y$ have ____ as a factor.	3
4	As in Chapter 12, we call 3 a <u>common</u> ______ of $3x$ and $3y$.	common factor

	$5xy$ is the product of 5, x, and y.	
5	$5yz$ is the product of 5, ___, and z.	y
	The common factors of $5xy$ and $5yz$ are 1, 5, y, and $5y$. As in Chapter 12, we call $5y$ the	
6	<u>greatest</u> ______ ______ of $5xy$ and $5yz$.	greatest common factor
7	Recognizing that ______ is the greatest common factor of $5xy$ and $5yz$ makes it easy to apply the distributive property to $5xy + 5yz$.	$5y$
8	$5xy + 5yz =$ ______	$5y(x + z)$

	In factoring $5xy + 5yz$ we might instead have used the following steps:	
9	$5xy + 5yz = 5(xy + \_\_\_\_\_)$	yz
	(5 is a common factor of $5xy$ and $5yz$.)	
	As a second step we observe that	
10	$(xy + yz) =$ ______	$y(x + z)$
11	(y is a ______ ______ of xy and yz.)	common factor
	So that $5xy + 5yz = 5(xy + yz)$	
	$= 5(y(x + z))$	
12	$= 5y(\_\_\_\_\_)$	$5y(x + z)$

Comparing the results in Items 8 and 12, we observe that we obtain the same complete factorization, whether we use one step or two. This may remind you of the Fundamental Theorem of Arithmetic. You will observe as we go along that there are a number of similarities between the properties of integers and those of polynomials.

The preceding example suggests that when we wish to use the distributive property to factor an indicated sum it is helpful to think about the common factors of the terms of the sum. In fact, the most helpful thing to do is to think about the greatest common factor.

13	The greatest common factor of $4t^2$ and $6t^4$ is ______.	$2t^2$
14	We can thus apply the distributive property to write: $4t^2 - 6t^4 =$ ______.	$2t^2(2 - 3t^2)$
	If you had trouble with Item 13, complete Items 15 to 18. If not, go to Item 19.	

	Consider $4t^2$ and $6t^4$.	
15	We note that $4 = 2^2$ and $6 = 2 \cdot$ ____.	$2 \cdot 3$
16	The greatest common factor of 4 and 6 is ______.	2
	t^2 is a factor of t^2.	
17	Also, since $t^4 = t^2 \cdot$ ______, t^2 is a factor of t^4.	$t^2 \cdot t^2$
18	Hence the greatest common factor of t^2 and t^4 is ______.	t^2
	Thus $2t^2$ is the greatest common factor of $4t^2$ and $6t^4$.	

19	Remember that $x(1) =$ _____.	x
20	Thus $xy + x = xy + x(1) =$ ___(______).	$x(y + 1)$

	Consider $6a^3b^2 - 3a^2b^3$.	
21	The greatest common factor of $6a^3b^2$ and $3a^2b^3$ is ______.	$3a^2b^2$
22	$6a^3b^2 - 3a^2b^3 =$ ______	$3a^2b^2(2a - b)$

23	Of the following, which factorization is complete? [A] $6s^2t - 3stu = 3st(2s - u)$ [B] $6s^2t - 3stu = 3s(2st - tu)$ [C] $6s^2t - 3stu = 3(2s^2t - stu)$

[A], [B], [C] are all true sentences, but only [A] gives the complete factorization. You should have chosen [A].

Factor completely.

24	$6x^2z - 3xyz =$ ______	$3xz(2x - y)$
25	$4x^2y^2 - 3xy + x =$ ______	$x(4xy^2 - 3y + 1)$
26	$-x^3y^2 + 2x^2y^2 + xy^2$ ______	$xy^2(-x^2+2x+1)$, or $-xy^2(x^2-2x-1)$

Although in Item 26 above you could have written either

$$-x^3y^2 + 2x^2y^2 + xy^2 = xy^2(-x^2 + 2x + 1)$$

$$\text{or} \quad -x^3y^2 + 2x^2y^2 + xy^2 = -xy\ (x^2 - 2x - 1),$$

the second form is usually preferred.

27	Which polynomial is factored correctly? [A] $2bx + 2 = 2b(x + 2)$ [B] $6a^3 - 9ab = 3a(2a^2 - 3b)$ [C] $4x^2y + 4x = 4x(xy + x)$

$2b(x + 2) = 2bx + 4b$ and $2bx + 4b \neq 2bx + 2$.
$4x(xy + x) = 4x^2y + 4x^2$ and $4x^2y + 4x^2 \neq 4x^2y + 4x$.
We see, therefore, that [A] and [C] are not examples of correctly factored polynomials. [B] illustrates an example of a correctly factored polynomial, since $3a(2a^2 - 3b) = 6a^3 - 9ab$.

In general, you will find it worthwhile to verify your factoring as in the response for Item 27.

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Let us consider some more complicated examples of factoring.

Consider the polynomial $(x - 1)t + (x - 1)3$. This

28 polynomial is the indicated sum of ______ $\underset{\text{(how many)}}{}$ terms. — two

29 Each term has ______ as a factor. — $x - 1$

We recognize that we can apply the distributive property.

30 $$ab + ac = \underline{\qquad}$$ — $a(b + c)$

That is,

31 $$\begin{array}{l} a \cdot b + a \cdot c = \boxed{a\ (b + c)} \\ \downarrow \\ (x - 1)t + (x - 1)3 = \boxed{(x - 1)(\qquad)}\ . \end{array}$$ — $\begin{array}{c} a\ (b+c) \\ \downarrow \\ (x-1)(t+3) \end{array}$

Note: Copy and complete the boxed material.

Thus $(x - 1)t + (x - 1)3 = (x - 1)(t + 3)$.

32 Similarly, the ______ $\underset{\text{(how many)}}{}$ terms of — two

$$(u + v)x - (u + v)y$$

33 have the common factor ______. — $u + v$

$$\begin{array}{l} a \quad b - a \quad c \\ \downarrow \\ (u + v)x - (u + v)y \end{array}$$

34 $(u + v)x - (u + v)y = \underline{\qquad}$ — $(u + v)(x - y)$

35 $x(x + 2) + 3(x + 2) = \underline{\qquad}$ — $(x + 3)(x + 2)$

(If you had trouble, complete Item 36. If not, go on to Item 37.)

36 $$\begin{array}{l} a \quad c + b \quad c = \boxed{(a + b)\ c} \\ \downarrow \\ x(x + 2) + 3(x + 2) = \boxed{(\qquad)(\qquad)} \end{array}$$ — $\begin{array}{c} (a+b)\ c \\ \downarrow \\ (x+3)(x+2) \end{array}$

	Factor each of the following:	
37	$a(x - 1) + (3x - 3) =$ ______	$(a + 3)(x - 1)$
38	$(a - b)a + (a - b)b =$ ______	$(a - b)(a + b)$
39	$x(4x - y) - y(4x - y) =$ ______	$(x - y)(4x - y)$
40	$3x(x + y) - 5y(x + y) + (x + y) =$ ______	$(3x-5y+1)(x+y)$
41	$r(u + v) - (u + v)s =$ ______	$(r - s)(u + v)$
42	$(a + b + c)x - (a + b + c)y =$ ______	$(a+b+c)(x-y)$

	Look carefully at this one: $z(x - y) + 3(y - x)$	
43	$x - y \underset{(=,\neq)}{\underline{\quad\quad}} y - x$	$\neq$
44.	However, $x - y \underset{(=,\neq)}{\underline{\quad\quad}} -(y - x)$	$=$
	Thus we can write:	
45	$z(x - y) + 3(y - x) = \underline{z(x - y) \quad\quad 3(x - y)}$	$z(x-y)-3(x-y)$
46	$= \underline{(\quad\quad)(\quad\quad)}$	$(z - 3)(x - y)$

	In the same way, factor completely:	
47	$a(x - 3) - b(3 - x) = \underline{a(x - 3) + b(\quad\quad)}$	$a(x-3)+b(x-3)$
48	$=$ ______	$(a + b)(x - 3)$
49	$2a(3x - 2y) + 3b(2y - 3x) =$ ______	$(2a-3b)(3x-2y)$ or $(3b-2a)(2y-3x)$

We have seen that in applying the distributive property, as in the foregoing examples, we begin by looking for common factors of the terms. This means, of course, that we must look at the factors of each term.

	For example, consider $5(z - 3) + (z^2 - 3z) = 0.$	
50	Note that $z^2 - 3z = \underline{z(\quad\quad)}.$	$z(z - 3)$
51	Thus $5(z - 3) + (z^2 - 3z) = 5(z - 3) + \underline{z(\quad\quad)}$	$z(z - 3)$
52	$= \underline{(\quad\quad)(\quad\quad)}$	$(5 + z)(z - 3)$

53	Factor: $(x - 1)(x + 2) + (x - 2)(x + 2)$. $(x-1)(x+2)+(x-2)(x+2) = ((x-1)+(x-2))(x+2)$ $= \_\_\_\_$	$(2x - 3)(x + 2)$
54	The common factor of $(x + 3)^2$ and $(x + 3)$ is ____.	$x + 3$
	Hence:	
55	$(x + 3)^2 - 2(x + 3) = ((\quad) - 2)(x + 3)$	$((x+3)-2)(x+3)$
56	$= (x + 1)(\quad)$	$(x + 1)(x + 3)$
	Factor completely:	
57	$(x + y)(u - v) + (x + y)v = \_\_\_\_$	$(x + y)u$
58	$(a + b + c)(x + y) - (a + b + c)y = \_\_\_\_$	$(a + b + c)x$
59	$144x^2 - 216s + 180y = \_\_\_\_$	$36(4x^2 - 6x + 5y)$
60	Use the same method to write a simpler name for $(r - s)(a + 2) + (s - r)(a + 2) = \_\_\_\_$.	0

We have had so much practice in factoring that you may be thinking that with skill and practice all polynomials can be factored. This is not the case.

61 Of the following, which cannot be factored over the integers?

M. $xy + yz + xz$ P. $12x + 15y$

N. $12x + 11y$ Q. $x(x - 2) + (x - 2)$

[A] M, N, and Q
[B] M, P, and Q
[C] M and N
[D] P and Q

$12x + 15y = 3(4x + 5y)$
$x(x - 2) + (x - 2) = (x + 1)(x - 2)$. (Did you remember that $x - 2 = (1)(x - 2)$?
M and N cannot be factored. Hence the correct choice is [C].

Just as we found many uses for the factored form of an integer, so we shall find many applications for the factored form of a polynomial. One of the more important uses is in solving equations.

62	Suppose we were asked to solve the equation $$5(z - 3) + (z^2 - 3z) = 0.$$ We have seen that the factored form of $5(z - 3) + (z^2 - 3z)$ is ______. (Look at Items 50 to 52 if you have trouble.)	$(5 + z)(z - 3)$
	Let us examine, then, the equation $$(5 + z)(z - 3) = 0$$	
63	The left side of this equation is a product, and the right side is ______.	0
	We know that the product of two real numbers is 0 if and only if one factor is 0.	
64	Hence the open sentence "$(5 + z)(z - 3) = 0$" is equivalent to the compound open sentence "$5 + z = 0$ or ______".	$z - 3 = 0$
65	The truth set of "$5 + z = 0$ or $z - 3 = 0$" is ______.	$\{-5, 3\}$
	Therefore, $\{-5,3\}$ is also the truth set of $$5(z - 3) + (z^2 - 3z) = 0.$$ We shall examine further examples of using the factored form of polynomials to solve polynomial equations as we proceed.	

You have now seen many examples in which it is possible to find a common factor for the terms of an indicated sum. In such examples, factoring is easy. Consequently, if you wish to factor a polynomial it is wise to begin by looking for a common factor. If you find one, you can proceed at once to apply the distributive property.

Suppose you don't? You cannot conclude at once that the polynomial cannot be factored over the integers.

	Let us consider, for example, the polynomial $$a^2 + ac + bc + ab.$$	
66	This is an indicated sum of ______ terms.	4
67	These terms ________ (do, do not) have a common factor.	do not
	However, we can group these terms: $a^2 + ac + bc + ab = (a^2 + ac) + (bc + ab)$.	
	Then we see	
68	$a^2 + ac =$ ______	$a(a + c)$
69	$bc + ab =$ ______	$b(a + c)$
	$a^2 + ac + bc + ab = (a^2 + ac) + (bc + ab)$	
	$= a(a + c) + b(a + c)$	
70	$=$ ______	$(a + b)(a + c)$

The preceding example illustrates the method called factoring by grouping terms. Sometimes this method is easy to use. Sometimes, however, it takes skill and ingenuity to see how the terms should be arranged and grouped. You may find that a polynomial which can be factored by grouping terms can also be factored by other methods which will be discussed in later sections of this chapter.

	Let us factor $x^2 + 4x + 3x + 12$ by grouping the terms.	
71	$x^2 + 4x + 3x + 12 = (x^2 + 4x) + (\quad\quad)$	$(x^2+4x)+(3x+12)$
72	$= x(\quad\quad) + 3(\quad\quad)$	$x(x+4) + 3(x+4)$
73	$=$ ______	$(x + 3)(x + 4)$

Suppose that in the preceding problem we had used the commutative property and grouped as follows.

$$x^2 + 4x + 3x + 12 = (x^2 + 12) + (4x + 3x)$$

In this form there is no common factor in the two terms. Thus we see the importance of choosing the proper grouping.

If you would like to try your hand at some more examples of factoring by grouping, complete Items *74 to *88.

* Consider the true sentence

$$2st + 6 - 3s - 4t = 2(st + 3) - (3s + 4t).$$

Although this sentence is true, it does not help us in factoring.

Using the commutative property let us write

$$2st + 6 - 3s - 4t = 2st - 3s + 6 - 4t$$

*74 $= s(\quad) - 2(\quad)$ — $s(2t-3)-2(2t-3)$

*75 $= (\quad)(\quad)$ — $(s - 2)(2t - 3)$

Factor completely.

*76 $ax + 2a + 3x + 6 =$ ______ — $(a + 3)(x + 2)$

*77 $3rs - 3s + 5r - 5 =$ ______ — $(3s + 5)(r - 1)$

*78 $5x + 3xy - 3y - 5 =$ ______ — $(x - 1)(5 + 3y)$

*79 $p^2 - mq - pq + mp =$ ______ — $(p + m)(p - q)$

*80 $ux + vx + uy + vy =$ ______ — $(u + v)(x + y)$

*81 $2ab + a^2 + 2b + a =$ ______ — $(a + 1)(2b + a)$

*82 $x^2 - 8x + x - 8 =$ ______ — $(x - 8)(x + 1)$

Suppose we were asked to factor $x^2 + 7x + 12$. As it stands we cannot apply our grouping technique to this polynomial.

*83 However, $7x = 4x +$ ______ — $3x$

so $x^2 + 7x + 12 = x^2 + 4x + 3x + 12$

*84 $= x(\quad) + 3(\quad)$ — $x(x+4)+3(x+4)$

*85 $=$ ______ — $(x + 3)(x + 4)$

Factor completely.

*86 $x^2 + 5x + 6 =$ ______ — $(x + 3)(x + 2)$

16-3. Difference of Squares

	We have seen already that for any two real numbers a and b,	
1	$(a + b)(a - b) = \underline{a(\quad) + b(\quad)}$	$a(a-b)+b(a-b)$
2	$= a^2 - ab + \_\_\_\_$	$ba-b^2$, or $ab-b^2$
3	$= \_\_\_\_$	$a^2 - b^2$

The sentence

$$(a + b)(a - b) = a^2 - b^2$$

is thus a simple consequence of the distributive property. It can be stated in words: The product of the <u>sum</u> and the <u>difference</u> of any two real numbers is equal to the difference of their squares.

4	$a + b$ is the ____ of a and b.	sum
5	$a - b$ is the ____ of a and b.	difference
6	$a^2 - b^2$ is the ____ of the squares of a and b.	difference
7	Since $(a + b)(a - b) =$ ____ for any real numbers	$a^2 - b^2$
8	a and b, we may say that the product of the ____	sum
	and the difference of any two real numbers is equal	
9	to the ____ of their ____.	difference, squares

10	The sum of $2x$ and $3y$ is written ____.	$2x + 3y$
11	The difference of $2x$ and $3y$ is written ____.	$2x - 3y$
	The product of the sum and difference of $2x$ and $3y$	
12	is written $(\quad)(\quad)$.	$(2x+3y)(2x-3y)$
	Applying the same pattern we may write	
	$(2x + 3y)(2x - 3y) = (2x)^2 - (3y)^2$	
13	$= \_\_\_\_$	$4x^2 - 9y^2$

In the last item you needed to write $(2x)^2$ and $(3y)^2$ in simple form. If you need review in using exponents in this way, do Items 14 to 19. If not, go to Item 20.

14	$(4x)^2 =$ ______	$16x^2$
15	$(7a^2)^2 =$ ______	$49a^4$
16	$(a^2b)^2 =$ ______	a^4b^2
17	$64x^2 = (\quad)^2$	$(8x)^2$
18	$36a^2b^2 = (\quad)^2$	$(6ab)^2$
19	$1 = (\quad)^2$	$(1)^2$

Using the same technique as in Item 13, write the following indicated products as indicated sums:

20	$(a - 2)(a + 2) =$ ______	$a^2 - 4$
21	$(2x - y)(2x + y) =$ ______	$4x^2 - y^2$
22	$(mn + 1)(mn - 1) =$ ______	$m^2n^2 - 1$
23	$(a^2 + b^2)(a^2 - b^2) =$ ______	$a^4 - b^4$
24	$(x - a)(x + a) =$ ______	$x^2 - a^2$

In Items 20-24 we have used the statement

$$a^2 - b^2 = (a + b)(a - b)$$

to change a product of two numbers into a certain related difference of squares.

Often we use the same statement to write a difference of two squares as a product.

Which form is preferable, difference or product, depends on the use we have in mind. In this chapter we are primarily concerned with <u>factoring</u>, so we wish to express polynomials as <u>products</u> whenever we can.

In Item 13 we found:

$$(2x + 3y)(2x - 3y) = 4x^2 - 9y^2.$$

If we are asked, on the other hand, to factor the polynomial $4x^2 - 9y^2$, we simply write:

25 $$4x^2 - 9y^2 = (\quad)(\quad)$$ $(2x+3y)(2x-3y)$

We are applying the fact that

26 $$(2x)^2 - (3y)^2 = (2x + 3y)(\_\_\_\_).$$ $2x - 3y$

Knowing that $a^2 - b^2 = (a + b)(a - b)$, we can always factor a polynomial if we can first write it as a difference of squares.

Thus to factor $9m^2 - 16n^2$, we recognize:

27 $9m^2 - 16n^2 = (\quad)^2 - (\quad)^2$ — $(3m)^2 - (4n)^2$

28 $= (\quad)(\quad)$ — $(3m+4n)(3m-4n)$

29 Similarly, $4x^2 - 25 = (\quad)(\quad)$. — $(2x+5)(2x-5)$

Using the sentence $a^2 - b^2 = (a + b)(a - b)$ as a model, factor the following:

30 $x^2 - 16 = (x + 4)(\quad)$ — $(x + 4)(x - 4)$

31 $a^2 - 9 = (\quad)(a - 3)$ — $(a + 3)(a - 3)$

32 $t^2 - 4 = (\quad)(\quad)$ — $(t + 2)(t - 2)$

33 $4x^2 - 1 =$ ____ — $(2x + 1)(2x - 1)$

34 $m^2 - 4 =$ ____ — $(m + 2)(m - 2)$

35 $9 - y^2 =$ ____ — $(3 + y)(3 - y)$

36 Below are two different procedures for factoring $16x^2 - 4y^2$ over the integers. Which procedure is simpler?

[A] $16x^2 - 4y^2 = 4(4x^2 - y^2)$

$= 4(2x + y)(2x - y)$

[B] $16x^2 - 4y^2 = (4x + 2y)(4x - 2y)$

$= 2(2x + y)\cdot 2(2x - y)$

$= 4(2x + y)(2x - y)$

If you noticed that one method takes only two steps, while the other takes three, you probably chose [A]. By recognizing first the common factor 4, we can factor completely in just two steps. In general, it is wise to look first for a factor which is common to every term.

In a similar manner we can factor such expressions as $8y^2 - 18$.

37 Note that: $8y^2 - 18 = 2(\quad)$ — $2(4y^2 - 9)$

38 $= 2(\quad)(\quad)$ — $2(2y+3)(2y-3)$

Factor the following expressions:

39 $24y^2 - 6z^2 =$ ____ — $6(2y+z)(2y-z)$

40 $1 - n^2 =$ ____ — $(1 + n)(1 - n)$

41 $25y^2 - 9 =$ ____ — $(5y+3)(5y-3)$

42 $25a^2 - b^2c^2 =$ ____ — $(5a+bc)(5a-bc)$

43 $20s^2 - 5 =$ ____ — $5(2s+1)(2s-1)$

44 $49x^4 - 1 =$ ____ — $(7x^2+1)(7x^2-1)$

Consider $x^2 - 2$.

45 2 ____ (is, is not) the square of an integer. — is not

$x^2 - 2$ is <u>not</u>, in fact, factorable over the integers.

Consider, now, $x^2 + 4$.

46 $x^2 + 4$ is a ____ (sum, difference) of two squares. — sum

Again, $x^2 + 4$ is <u>not</u> factorable over the integers.

Factor the following over the integers, if possible. If the polynomial cannot be factored over the integers, write "not factorable over the integers".

47 $x^2 - 4 =$ ____ — $(x + 2)(x - 2)$

48 $x^2 + 16 =$ ____ — not factorable over the integers

49 $x^2 - 3 =$ ____ — not factorable over the integers

50 $x^4 - 16 =$ ____ — $(x^2+4)(x+2)(x-2)$

(Hint: There are 3 factors in the complete factorization.)

51	$x^4 - 4 =$ ______	$(x^2+2)(x^2-2)$
52	$3x^2 - 3 =$ ______	$3(x+1)(x-1)$
53	$16x^4 - 1 = (\_\_\_\_)(2x + 1)(2x - 1)$	$(4x^2 + 1)$

We may apply our method of factoring the difference of two squares to such polynomials as

$$(a - 1)^2 - 1.$$

	$(a - 1)^2 - 1 = (a - 1)^2 - 1^2$	
	$= \big((a - 1) + 1\big)\big((a - 1) - 1\big)$	
54	$=$ ______	$a(a - 2)$
55	$(a - 3)^2 - 16 = \big((a - 3) + 4\big)\big((\quad) - \quad\big)$	$\big((a - 3) - 4\big)$
56	$=$ ______	$(a + 1)(a - 7)$
57	$(m + n)^2 - (m - n)^2 =$ ______	$4mn$
58	$(m + n)^2 - (m + n)^2 =$ ______	0
*59	$(x^2 - y^2) - (x - y) =$ ______	$(x + y - 1)(x - y)$

As we saw in the preceding section, factoring may be applied in solving certain polynomial equations.
Let us solve the equation $x^2 - 9 = 0$.
We can reason:

	If $x^2 - 9 = 0$ for some x,	
60	then $(x + 3)(\quad) = 0$ for the same x.	$(x+3)(x-3)=0$
	$(x + 3)(x - 3) = 0$ if and only if either $x + 3 = 0$	
61	or $x - 3 =$ ____.	0
62	If $x + 3 = 0$, then $x =$ ____.	-3
63	If $x - 3 = 0$, then $x =$ ____.	3
	We note that "$x^2 - 9 = 0$" and "$x = 3$ or $x = -3$" are equivalent open sentences.	
64	Thus, the truth set of $x^2 - 9 = 0$ is ______.	$\{3,-3\}$

	We know another way of solving the equation $x^2 - 9 = 0$. We need only note that	
65	"$x^2 - 9 = 0$" and "$x^2 = $ ____"	9
	are equivalent equations.	
66	We see at once that the truth set of $x^2 = 9$ is {____}.	$\{3, -3\}$
	Either method leads to the same truth set, which is certainly what would be expected.	

	Solve each equation. Try in at least some of them to use both methods--that illustrated in Items 60 to 64 and that of Items 65 to 66. The two methods lead, of course, to the same result.	
67	$9r^2 - 4 = 0$ ____	$\{\frac{2}{3}, -\frac{2}{3}\}$
68	$5x^2 - 20 = 0$ ____	$\{2, -2\}$
69	$75a^2 - 3 = 0$ ____	$\{\frac{1}{5}, -\frac{1}{5}\}$
70	$2x^2 = 8$ ____	$\{-2, 2\}$
71	$4t^3 - t = 0$ ____	$\{0, -\frac{1}{2}, \frac{1}{2}\}$
72	$(s + 2)^2 - 9 = 0$ ____	$\{1, -5\}$
73	$x^2 + 4 = 0$ ____	$\emptyset$, (since $x^2+4>0$ for all real numbers x)

Item 73 is interesting. Notice that our statement that we cannot factor the sum of x^2 and 4 is consistent with the fact that there is no real number x for which $x^2 + 4$ is 0.

In this section we have used in factoring the fact that if a and b are any real numbers

$$a^2 - b^2 = (a + b)(a - b).$$

This fact has many other applications, one of which will be noted in Items *74 to *100.

*	Consider the product of the sum and difference of the two numbers $\sqrt{2}$ and 3. $$(\sqrt{2} + 3)(\sqrt{2} - 3)$$	
*74	$\sqrt{2}$ is an <u>irrational</u> number. Hence, $\sqrt{2} + 3$ and $\sqrt{2} - 3$ are both ______ numbers.	irrational
	$(\sqrt{2} + 3)(\sqrt{2} - 3) = (\sqrt{2})^2 - (3)^2$	
*75	$= 2 - \_\_\_\_$	$2 - 9$
*76	$= \_\_\_\_$	-7
*77	The product, -7, is a(n) ______ (rational, irrational) number.	rational
	Consider $(\sqrt{3} + \sqrt{5})(\sqrt{3} - \sqrt{5})$.	
*78	$(\sqrt{3} + \sqrt{5})(\sqrt{3} - \sqrt{5}) = (\quad)^2 - (\quad)^2$	$(\sqrt{3})^2 - (\sqrt{5})^2$
*79	$= \_\_\_\_ - 5$	$3 - 5$
*80	$= \_\_\_\_$	-2
*81	The product, -2, is a ______ (rational, irrational) number.	rational

* From the exercises which you have done, it should be evident that, given an indicated product of the form $(a + b)(a - b)$, the product <u>may</u> be rational even when either a or b is irrational, or even if both are irrational.

In fact, the product of the sum and difference of two real numbers a and b which are either rational or at most square roots of rational numbers is itself a rational number.

	We shall use this property to rationalize the denominator in $\frac{1}{5 - \sqrt{3}}$.	
	$\frac{1}{5 - \sqrt{3}} = \frac{1}{5 - \sqrt{3}} \cdot \frac{5 + \sqrt{3}}{5 + \sqrt{3}}$ by the multiplication property of 1.	
*82	$= \frac{\boxed{}}{(5 - \sqrt{3})(5 + \sqrt{3})}$	$5 + \sqrt{3}$
	$= \frac{5 + \sqrt{3}}{5^2 - (\sqrt{3})^2}$	
*83	$= \_\_\_\_$	$\frac{5 + \sqrt{3}}{22}$
	Thus, $\frac{1}{5 - \sqrt{3}} = \frac{5 + \sqrt{3}}{22}$.	
*84	We have completed "rationalizing the ______" of the fraction.	denominator

* As you will recall from Chapter 15, we sometimes find rationalizing the denominator very convenient. You can find in tables, for example, that $\sqrt{3}$ is approximately 1.732. If you want to approximate $\frac{1}{5 - \sqrt{3}}$ by a decimal, it saves arithmetic to notice that

$$\frac{1}{5 - \sqrt{3}} = \frac{5 + \sqrt{3}}{22}$$

Test it for yourself if you don't see why.

*	Rationalize the denominator in each of the following:	
*85	$\frac{2}{5 + \sqrt{2}} = \frac{2}{5 + \sqrt{2}} \cdot \frac{5 - \sqrt{2}}{(\quad)}$	$\frac{5 - \sqrt{2}}{5 - \sqrt{2}}$
*86	$= \underline{\qquad}$	$\frac{2(5 - \sqrt{2})}{23}$
*87	$\frac{3 + \sqrt{5}}{3 - \sqrt{5}} = \underline{\qquad}$ $[(3 + \sqrt{5})(3 + \sqrt{5}) = 9 + 6\sqrt{5} + 5 = 14 + 6\sqrt{5} = 2(7 + 3\sqrt{5})]$	$\frac{7 + 3\sqrt{5}}{2}$
*88	$\frac{7}{\sqrt{11} - 3} = \underline{\qquad}$	$\frac{7(\sqrt{11} + 3)}{2}$
*89	$\frac{\sqrt{3}}{\sqrt{6} - \sqrt{5}} = \underline{\qquad}$	$3\sqrt{2} + \sqrt{15}$

* We find it easy to factor $a^2 - b^2$, which is the <u>difference</u> of two squares. In fact $a^2 - b^2 = (a + b)(a - b)$. But there is no way to factor the <u>sum</u> $a^2 + b^2$ over the integers.

It is natural to ask: What about

$$a^3 - b^3 \quad \text{and} \quad a^3 + b^3 \text{ ?}$$

The answers are easy to find.

*	Find the product:	
*90	$(a + b)(a^2 - ab + b^2) = a(a^2 - ab + b^2) + \underline{b(\qquad)}$	$b(a^2 - ab + b^2)$
*91	$= \underline{a^3 \qquad}.$	$a^3 + b^3$
	We see a neat factorization:	
	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.	

We can use this result,

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$, to factor $t^3 + 1$.

*92 We need only let $t = a$ and ______ b. | $1 = b$

From $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ we obtain:

*93 $t^3 + 1 = \underline{(\quad\quad)(\quad - \quad + \quad)}$. | $(t+1)(t^2-t+1)$

Factor $s^3 + 8$, using the result that

*94 $a^3 + b^3 = \underline{(a \quad\quad)(a^2 - ab + b^2)}$. | $(a+b)(a^2-ab+b^2)$

Let $a = s$ and $b = 2$.

*95 $s^3 + 8 = s^3 + (2)^3 =$ ______. | $(s+2)(s^2-2s+4)$

* Find the product:

$$(a-b)(a^2+ab+b^2) = a(a^2+ab+b^2) - b(a^2+ab+b^2)$$

*96 = ______ | $a^3 - b^3$

We have shown that $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

*97 To factor $a^3 - 8$ we write $a^3 - 8 = \underline{a^3 - (\quad)^3}$. | $a^3 - (2)^3$

*98 Thus, $a^3 - 8 = \underline{(a - \quad)(a^2 + 2a + \quad)}$. | $(a-2)(a^2+2a+4)$

*99 $8x^3 - 1 = \underline{(2x - \quad)(\quad\quad\quad)}$ | $(2x-1)(4x^2+2x+1)$

*100 $t^3 - 1 = \underline{(\quad\quad)(\quad\quad\quad)}$ | $(t-1)(t^2+t+1)$

Thus we see: It is possible to factor <u>both</u> the sum and the difference of two cubes.

16-4. Perfect Squares

If a polynomial is the product of two identical polynomials, it is said to be a *perfect square*.

	Since $(3mn)^2 = (3mn)(3mn)$ $= 9m^2n^2$	
1	we see that $9m^2n^2$ is a perfect ______.	square
	Here is a more-interesting example. $(a + b)^2 = (a + b)(a + b)$	
2	$= a(a + b) + b(\_\_\_\_)$	$b(a + b)$
3	$= a^2 + ab + ba + \_\_\_\_$	b^2
4	$= a^2 + \_\_\_ ab + b^2$	$a^2 + 2ab + b^2$
	Similarly $(a - b)^2 = (a - b)(a - b)$	
5	$= a(a - b) - b\_\_\_\_$	$b(a - b)$
6	$= a^2\_\_\_\_\_\_$	$a^2 - 2ab + b^2$

Since

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

the polynomials $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$ are both perfect squares. Each is the product of two identical polynomials.

In this section we will learn to use these patterns to recognize perfect squares.

7 Which of the following is not a perfect square?

[A] x^2 [C] $x^2 + 2xy + y^2$

[B] $25x^5y^4$ [D] $x^2 - 2xy + y^2$

We know that $x \cdot x = x^2$, $(x + y)(x + y) = x^2 + 2xy + y^2$, and $(x - y)(x - y) = x^2 - 2xy + y^2$. Thus, the polynomials in [A], [C], and [D] may be written as the product of two identical polynomials and are therefore perfect squares. $25x^5y^4$ is not a perfect square since it cannot be written as a product of two identical polynomials. [B] is the correct choice.

	We have noted that	
	$(a + b)^2 = a^2 + 2ab + b^2$.	
	This result may be expressed in words:	
	"The square of the sum of two numbers is the square of	
8	the first number plus ______ their product plus the	twice, or two times
9	______ of the second number."	square
	Square the following, using this pattern.	
10	$(10 + 3)^2 = 10^2 + 2(10)(3) +$ ____	3^2, or 9
	$= 169$	
11	$(20 + 5)^2 = 20^2 + \underline{2(\quad)(\quad)} + 5^2$	2(20)(5)
12	$=$ ______	625

	Square the following:	
13	$(x + 4)^2 = x^2 + \underline{2(\quad)(\quad)} + 4^2$	$2(x)(4)$
14	$=$ ______	$x^2 + 8x + 16$
15	$(1 + c)^2 = \underline{(\quad)^2 + 2(\quad)(\quad) + (\quad)^2}$	$1^2 + 2(1)(c) + c^2$
16	$=$ ______	$1 + 2c + c^2$
	Now see if you can write the product without writing down the middle step:	
17	$(2x + y)^2 =$ ______	$4x^2 + 4xy + y^2$
18	$(3a + 2b)^2 =$ ______	$9a^2 + 12ab + 4b^2$
19	$(3x + 5)^2 =$ ______	$9x^2 + 30x + 25$
20	$(2m + 5n)^2 =$ ______	$4m^2 + 20mn + 25n^2$
21	$(7a + 3b)^2 =$ ______	$49a^2 + 42ab + 9b^2$

*22 Here is an example which is a little more difficult.

$((x + 2) + y)^2 =$ ______

$x^2 + y^2 + 2xy + 4x + 4y + 4$

Now let us consider the polynomial

[illegible]

Is it [illegible] compare it with [illegible]

$x^2 + 6x$ [illegible]

as $x^2 + 2(x)(3) + 3^2$

and compare this form with the pattern:

$$\begin{array}{c}
a^2 + 2\,a\,b + b^2 = (a + b)^2 \\
\updownarrow \\
x^2 + 2(x)(3) + 3^2 = (x + 3)^2
\end{array}$$

We have used x for a and 3 for b. We may conclude that $x^2 + 6x + 9$ is a perfect square.

Is the polynomial $36x^2 + 12xy + y^2$ a perfect square?

Our procedure, of course, is to compare it with the

23 polynomial $a^2 +$ ______.

$a^2 + 2ab + b^2$

24 $36x^2 + 12xy + y^2$ may be written

$$\begin{array}{c}
(\quad)^2 + 2(6x)(y) + y^2 \\
\updownarrow \\
a^2 + 2\,_____ + b^2
\end{array}$$

See answer below.

(Note: Copy and complete boxed material)

$$\begin{array}{c}
(6x)^2 + 2(6x)(y) + y^2 \\
\updownarrow \\
a^2 + 2\,a\,b + b^2
\end{array}$$

25 The polynomial $36x^2 + 12xy + y^2$ ______ (is, is not) a perfect square, and we may write:

is

26 $36x^2 + 12xy + y^2 = ($ ______ $)^2$.

$(6x + y)^2$

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	Find the <u>perfect squares</u> among the following polynomials, and express them as squares. For any that are not perfect squares, write "not a perfect square".	
27	$x^2 + 2xy + y^2 =$ ______	$(x + y)^2$
28	$4x^2 + 4x - 1 =$ ______	not a perfect square
29	$x^2 + 6xy + 9y^2 =$ ______	$(x + 3y)^2$
30	$4x^2 + 4x + 1 =$ ______	$(2x + 1)^2$
31	$4x^2 + 12xy + 9y^2 =$ ______	$(2x + 3y)^2$

We also noted that $(a - b)^2 = a^2 - 2ab + b^2$. That is, $a^2 - 2ab + b^2$ is a perfect square since it may be written as the product of two identical polynomials. It is often convenient to compare polynomials with this pattern to test whether or not they are perfect squares.

	For example: $9x^2 - 12xy + 4y^2$	
32	can be written: $(\ \ \ )^2 - 2(3x)(2y) + (\ \ \ )^2$ and compared with: $a^2 - 2\ a\ b + b^2$.	$(3x)^2$, $(2y)^2$
	We may conclude from this test that the polynomial	
33	$9x^2 - 12xy + 4y^2$ ______ (is, is not) a perfect square.	is
	Comparing $9x^2 - 12xy + 4y^2$ with $a^2 - 2ab + b^2$, we	
34	identified a with ______	3x
35	and b with ______.	2y
36	$9x^2 - 12xy + 4y^2 =$ ______	$(3x - 2y)^2$

	Find the <u>perfect squares</u> among the following polynomials, and express them as squares. For any that are not perfect squares, write "not a perfect square".	
37	$x^2 - 2xy + y^2 =$ ______	$(x - y)^2$
38	$4x^2 - 4x + y^2 =$ ______	not a perfect square

39 [illegible] ______ — not a perfect square

40 [illegible] ______ — $(x + 4)^2$

41 $4x^2y^2 - 4xyz + z^2$ ______ — $(2xy - z)^2$

Express the following polynomials as squares:

42 $y^2 - 2yd + d^2$ ______ — $(y - d)^2$

43 $x^2y^2 + 2xyz + z^2$ ______ — $(xy + z)^2$

44 [illegible] ______ — $(3u + 4v)^2$

45 $(x + 1)^2 + 2(x + 1)y + y^2$ ______. — $((x + 1) + y)^2$ [or, simply, $(x + 1 + y)^2$]

46 $25z^2 - 40z + 16$ ______ — $(5z - 4)^2$

Suppose you are given a quadratic polynomial over the integers of the form $x^2 + px + q$.

(Note that in $x^2 + px + q$ the coefficient of x^2

47 is ______.) — 1

48 What must be true of p and q if the ______ polynomial $x^2 + px + q$ is a perfect square? — quadratic

If $x^2 + px + q$ is a perfect square, then we must have one or the other of these patterns:

$$\begin{array}{ccccc} x^2 & + & px & + & q \\ \updownarrow & & \updownarrow & & \updownarrow \\ a^2 & + & 2ab & + & b^2 \end{array} \qquad \text{or} \qquad \begin{array}{ccccc} x^2 & + & px & + & q \\ \updownarrow & & \updownarrow & & \updownarrow \\ a^2 & - & 2ab & + & b^2 \end{array}$$

49 In either case, we have x^2 as a^2 and q as ______. — b^2

50 Hence, the integer q must be the ______ of some integer b. — square

In $2ab$ or $-2ab$, if the a is identified with x, then

$$\begin{array}{ccccc} x^2 & + & px & + & q \\ \updownarrow & & \updownarrow & & \updownarrow \\ a^2 & + & 2ab & + & b^2 \end{array} \qquad \text{or} \qquad \begin{array}{ccccc} x^2 & + & px & + & q \\ \updownarrow & & \updownarrow & & \updownarrow \\ a^2 & - & 2ab & + & b^2 \end{array}$$

51 Hence, either $2b = p$ or ______ $= p$. — $-2b = p$

Thus we can see that $x^2 + 6x + 9$ is a perfect square.

53 This is because $9 = 3^2$ and $6 = 2(\_\_\_)$. — 2(3)

$m^2 + 8m + 16$ is a perfect square, since $16 = (\_\_\_)^2$ — $(4)^2$

54 and $8 = 2(\_\_\_)$. — 2(4)

Let us look at some more examples.

We know that $x^2 + 10x + 25$ is a perfect square, since

55 $25 = (\_\_\_)^2$ and $10 = 2 \times 5$. — $(5)^2$

56 $x^2 - 20x + 100$ is a perfect square since $100 = \_\_\_\_$ — $(10)^2$

and $-20 = -(2 \times 10)$.

57 $x^2 - 8x + 16$ is a perfect square since $16 = \_\_\_\_$ — 4^2

58 and $-8 = \_\_\_\_$. — $-(2 \times 4)$

To summarize we may say:

A quadratic polynomial $x^2 + px + q$, where p and q are integers, is a perfect square if and only if

1) q is the square of an integer b.

2) either $p = 2b$ or $p = -2b$.

If we saw "$x^2 + 12x + (\quad)$" we might ask what number we could put in the parentheses to make the resulting polynomial a perfect square.

59 Our discussion suggests that we put in the number $\_\_\_\_$. — 36

We see that $x^2 + 12x + 36$ is a perfect square, since

60 $36 = (\_\_\_)^2$ and $12 = \_\_\_\_$. — $(6)^2$ or 6^2, 2(6)

Fill the blank with a number which will make the resulting polynomial a perfect square.

61 $x^2 + 16x + \_\_\_\_$ — 64

62 $x^2 - 2x + \_\_\_\_$ — 1

63 $a^2 + \_\_\_\_ + 49$ — 14a

Complete the following to make true sentences

64 $x^2 + 2xy + \_\_\_\_ = (\_\_\_\_)^2$ — y^2, $(x + y)^2$

65 $c^2 + 4c + \_\_\_\_ = (\_\_\_\_)^2$ — 4, $(c + 2)^2$

66 $x^2 + 8x + \_\_\_\_ = (\_\_\_\_)^2$ — 16, $(x + 4)^2$

We are primarily interested just now in factoring polynomials over the integers. However, we should be aware that the ideas we are using have wide applicability.

The statements

$$a^2 - b^2 = (a + b)(a - b)$$

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

are true, of course, if a and b are <u>any</u> real numbers. Our use of these statements is not always restricted to the integers.

We observe, for example, that

67 $$\left(x + \frac{3}{2}\right)^2 = x^2 + \_\_\_\_\_\_$$ — $x^2 + 3x + \frac{9}{4}$

This illustrates once again the pattern

68 $$\begin{array}{c} (a + b)^2 = a^2 + \_\_\_\_\_\_ + b^2 \\ \left(x + \frac{3}{2}\right)^2 = x^2 + 2(x)(\_\_\_) + (\_\_\_)^2 \end{array}$$ — **See answer below.**

(Copy and complete.)

$$\begin{array}{c} (a + b)^2 = a^2 + 2\,a\,b + b^2 \\ \left(x + \frac{3}{2}\right)^2 = x^2 + 2(x)\left(\frac{3}{2}\right) + \left(\frac{3}{2}\right)^2 \end{array}$$

Fill in the blanks to make the following polynomials perfect squares.

69 $x^2 + 5x + (\_\_\_)^2$ — $\left(\frac{5}{2}\right)^2$

70 $x^2 - 3x + (\_\_\_)^2$ — $\left(\frac{3}{2}\right)^2$

71 $x^2 + x + (\_\_\_)^2$ [Hint: The coefficient of x is 1] — $\left(\frac{1}{2}\right)^2$

	Fill in the missing term to make each of the following a perfect square, and complete the sentence.	
72	$x^2 + 10x + \_\_\_\_ = (\_\_\_\_\_)^2$	$25,\ (x+5)^2$
73	$y^2 - y + \_\_\_\_ = (\_\_\_\_\_)^2$	$\frac{1}{4},\ (y - \frac{1}{2})^2$
74	$b^2 + \frac{2}{5}b + \_\_\_\_ = (\_\_\_\_\_)^2$	$\frac{1}{25},\ (b + \frac{1}{5})^2$
75	$x^2 - 7x + \_\_\_\_ = (\_\_\_\_\_)^2$	$\frac{49}{4},\ (x - \frac{7}{2})^2$
76	$x^2 - \frac{x}{2} + \_\_\_\_ = (\_\_\_\_\_)^2$	$\frac{1}{16},\ (x - \frac{1}{4})^2$
	[Hint: The coefficient of x is $-\frac{1}{2}$.]	

We call the process illustrated in Items 64 to 66 and Items 69 to 76 completing the square. This process will play an important part in our discussion of quadratic polynomials.

We will conclude this section by showing how recognizing perfect squares helps us in solving certain equations.

	For example, let us solve $4z^2 + 4z + 1 = 0$.	
77	Notice that the right side of this equation is ______.	0
78	Moreover, we recognize that $4z^2 + 4z + 1$ is a perfect ______.	square
79	We thus see that $4z^2 + 4z + 1 = 0$ $(\_\_\_\_\_)^2 = 0$ are equivalent equations.	$(2z + 1)^2$
	$(2z + 1)^2$ is a product of two identical factors. This product is 0 if and only if $2x + 1$ is 0.	
	Thus we have the chain of equivalent sentences: $4z^2 + 4z + 1 = 0$ $(2z + 1)^2 = 0$	
80	$\_\_\_\_\_\_ = 0$	$2z + 1 = 0$
81	The truth set for each is ______.	$\{-\frac{1}{2}\}$

	Solve	
82	$x^2 - 10x + 25 = 0$ Truth set: ______	{5}
83	$9x^2 + 12x + 4 = 0$ Truth set: ______	$\{-\frac{2}{3}\}$

16-5. Factoring by Completing the Square

By now you should be able to factor a polynomial which is the difference of two squares. You should also be able to construct polynomials which are perfect squares.

These techniques can be combined to factor certain polynomials which are not perfect squares.

	Consider, for example, the polynomial $x^2 + 6x + 5$.	
1	The polynomial $\underset{\text{(is, is not)}}{\underline{\qquad\qquad}}$ a perfect square.	is not
2	However, we know that $x^2 + 6x +$ _____ is a perfect square.	9
3	Moreover, $x^2 + 6x + 5 = x^2 + 6x + \overbrace{9 - \underline{\quad}}^{\text{zero}} + 5$	-9
4	$= (x^2 + 6x + 9) - \underline{\quad}$	-4
5	$= (\underline{\qquad})^2 - 4$	$(x + 3)^2 - 4$
6	We observe that $(x + 3)^2 - 4$ is the ______ of two squares. Hence, continuing:	difference
	$x^2 + 6x + 5 = (x + 3)^2 - 4$	
7	$= (x + 3 + 2)(\underline{\qquad\qquad})$	$(x + 3 - 2)$
	$= (x + 5)(x + 1)$.	
8	In Item 3 we added and subtracted _____. This is one step in the process of completing the square.	9

192

	Here is another example.	
9	Note that $x^2 + 4x - 5$ ______ a perfect square. (is, is not)	is not
10	But $x^2 + 4x +$ ____ is a perfect square.	+ 4
	Thus, we write:	
11	$x^2 + 4x - 5 = \underbrace{x^2 + 4x + 4}_{\text{perfect square}}\overbrace{- (\_\_\_\_)}^{\text{zero}} - 5.$	(4)
	$x^2 + 4x + 4 - 4 - 5 = (x^2 + 4x + 4) - 9 =$	
12	$(\_\_\_\_)^2 - \_\_\_\_$	$(x + 2)^2 - 9$
	$(x + 2 + 3)(x + 2 - 3)$	
13	____	$(x + 5)(x - 1)$

	To factor $x^2 - 8x + 7$ we note first that	
14	$x^2 - 8x +$ ____ is a perfect square.	16
	By adding 0, in the form of 16 - 16, we write:	
15	$x^2 - 8x + 7 = x^2 - 8x +$ ____ - ____ $+ 7$	16 - 16
16	$(x^2 - 8x +$ ____ $) - 9$	16
	$= (x - 4)^2 - 9$	
17	____	$(x - 7)(x - 1)$

	Factor the following by completing the square.	
	$x^2 - 6x + 8 = (x - 3)^2 - 1^2$ (add and subtract 9)	
18	= ____	$(x - 4)(x - 2)$
	$x^2 + 4x + 3 = (x + 2)^2 - 1^2$	
19	= ____	$(x + 3)(x + 1)$
20	$x^2 - 2x - 8 =$ ____	$(x - 4)(x + 2)$
21	$x^2 - 10x + 24 =$ ____	$(x - 6)(x - 4)$

[illegible] What [illegible] a perfect square?	
[illegible] the coefficient [illegible]	x
[illegible]	$\frac{1}{2}(3)$
[illegible] = (______)^2.	$(x + \frac{3}{2})^2$

[illegible]	
[illegible] we can write:	
[illegible] ______	$+ \frac{9}{4} - \frac{9}{4}$
______	$(x + \frac{3}{2})^2 - \frac{25}{4}$
[illegible] = ______	$(\frac{5}{2})^2$
[illegible] ______	$(x + \frac{3}{2} - \frac{5}{2})$
(______)(______)	$(x + 4)(x - 1)$

In [illegible] we [illegible]:

$$x^2 + 3x - 4 = (x + 4)(x - 1),$$

and we have factored the polynomial $x^2 + 3x - 4$ over the integers by completing the square. It is interesting to observe that in doing this we used rational numbers ([illegible]) that are not integers.

Factor over the integers.	
[illegible] ______	$(y + 3)(y - 7)$
[illegible] ______	$(y + 2)(y - 1)$

We have seen a number of examples of quadratic polynomials which can be factored over the integers by completing the square. In each case we found that after we completed the square we were able to express the polynomial as a difference of two squares. As you may suspect, things are not always quite so simple. Let us look at some further examples.

	If we attempt to factor $x^2 + 2x + 2$ by completing the square we obtain:	
32	$x^2 + 2x + 2 = \underline{(x + 1)^2 + \quad\quad}$.	$(x + 1)^2 + 1$
33	We ______ (have, have not) obtained a result which is the difference of two squares.	have not
34	Indeed, $(x + 1)^2 + 1$ is the ______ of two squares. We cannot factor this polynomial.	sum

	Now let us consider $x^2 + 2x - 2$. In this case completing the square leads to:	
35	$x^2 + 2x - 2 = \underline{(x + 1)^2 \quad\quad\quad}$.	$(x + 1)^2 - 3$
36	This is a ______ (sum, difference) of two terms, $(x + 1)^2$	difference
37	and ____.	3
38	Is there an <u>integer</u> whose square is 3? ______	No
39	Hence we ______ (can, cannot) factor $(x + 1)^2 - 3$ over the integers.	cannot

You might like to see how this relates to some of the things we will do later. Items *40 to *42, which are optional, deal with this.

*40	Is there a <u>real</u> number whose square is 3? ______	Yes
*41	We note that $(\sqrt{3})^2 =$ ____.	3
	Hence $x^2 + 2x + 2 = (x + 1)^2 - 3$	
	$= (x + 1)^2 - (\sqrt{3})^2$	
*42	$= (x + 1 + \sqrt{3})(\underline{\quad\quad\quad})$.	$(x + 1 - \sqrt{3})$

[illegible]

[illegible] number [illegible].

[illegible]

1) [illegible] the distributive property.

2) If the polynomial is the difference of two squares, we may [illegible] pattern

$$[illegible] = (a - [illegible])(a + [illegible]).$$

3) If we have a polynomial with the pattern [illegible] or $a^2 - [illegible]$, then the polynomial is a perfect square.

4) Finally, if we have a polynomial of the form [illegible] completing the square. [illegible], however, that [illegible] enable us to factor the polynomial over the integers.

[illegible] over the integers.

Factor each of the following over the integers if possible. If the polynomial cannot be factored, write "not possible". Answers, and some suggestions, are on page xix.

1. [illegible] ________ 6. [illegible] ________
2. $x^2 - 2x -$ [illegible] ________ 7. [illegible] ________
3. [illegible] ________ 8. [illegible] ________
4. [illegible] ________ 9. [illegible] ________
5. [illegible] ________

Once again recall the following property of real numbers. For [illegible] real numbers, [illegible] if and only if [illegible] or ________.

10 [illegible]

Because of this property, as we have already seen, factoring plays an important role in solving certain equations.

	In order to solve $x^2 = 4x - 4$ we may first write the equivalent sentence	
11	$x^2 - 4x + 4 =$ ____.	0
	This equation has 0 on one side; hence we may apply the property stated in Item 10.	
	$x^2 = 4x - 4$	
12	$x^2 - 4x + 4 =$ ____	0
13	$\underline{(x \qquad)^2} = 0$	$(x - 2)^2$
	$x - 2 = 0$	
14	are all ______ sentences.	equivalent
15	Hence the truth set of $x^2 = 4x - 4$ is ______.	$\{2\}$

Solve. Answers are on page xix.

16. $x^2 + 2x = 0$
17. $x^2 - 25 = 0$
18. $(y + 1)^2 - 16 = 0$
19. $y^2 + 6y + 12 = 0$
20. $x^2 - 4x = 5$

Chapter 17

QUADRATIC POLYNOMIALS

17-1. Factoring by Inspection

We have already learned that we can use the method of completing the square in factoring quadratic polynomials of the form $x^2 + px + q$. Sometimes we are able to factor such polynomials over the integers, but sometimes we cannot do this.

Now we are going to study a second approach to the problem of factoring such polynomials over the integers. It requires ingenuity, but it is often quicker than completing the square.

We have found that examining carefully the process for writing an indicated product as a sum helps us recognize patterns useful in factoring.

Let us begin, then, by considering as an example the product $(2x + 5)(3x + 7)$.

1	$(2x + 5)(3x + 7) = 2x(3x + 7) +$ _______	$5(3x + 7)$
2	$= 6x^2 +$ ___ $x + 15x + 35$	$14x$
3	$= 6x^2 +$ ___ $x + 35$	$29x$
4	In Items 1 and 2 we have applied the _______ property.	distributive

We can show the work of Items 1, 2, 3 in a diagram as follows:

Step 1	Step 2	Step 3
$(2x + 5)(3x + 7)$	$(2x + 5)(3x + 7)$	$(2x + 5)(3x + 7)$
	+	
$6x^2$	$14x + 15x$	35
	$29x$	

Look carefully at this diagram as you complete Items 5 to 10.

	In Item 2 we saw that $(2x + 5)(3x + 7) = 6x^2 + 14x + 15x + 35$. Each of these terms is the product of one term in $(2x + 5)$ and one term in $(3x + 7)$.	
5	Thus $6x^2$ is the product of the $2x$ in $(2x + 5)$ and the _______ in $(3x + 7)$.	$3x$
6	We indicate this in the diagram in Step 1. The $2x$ and the $3x$, which are multiplied to give _______, are connected by a curve.	$6x^2$

In Step [illegible] [illegible] ______. [illegible]

[illegible] ______. [illegible]

In Step [illegible] we identify [illegible] ______.

Let us compute the product $(x + [illegible])(x + [illegible])$ following this same procedure.

Step 1 [illegible] Step 2 [illegible] Step 3 [illegible]

[illegible]

Thus we can determine [illegible] the product $(x + [illegible])(x + [illegible]) = x^2 + 10x + [illegible]$.

In Step [illegible], we see that [illegible] outside terms in the product. [illegible] the outside product.

Similarly, [illegible] inside [illegible] we call it the ______ product. inside

11 The sum of [illegible] and [illegible] is ______. 10x

We sometimes call [illegible] the middle term of the [illegible]nomial $x^2 + 10x + [illegible]$.

Let us look at one more example.

Step 1

12 $(x + [illegible])(x + [illegible])$ ______ + ... x^2

Step 2

13 $(x + [illegible])(x + [illegible]) = x^2 +$ ______ + ... 8x

Step 3

14 $(x + [illegible])(x + [illegible]) = x^2 + [illegible]x +$ ______ 15

With practice, you will be able to think through the steps shown in the diagram without drawing the curves.

15 Which of the following indicated products when written as a sum has $-5x$ as the middle term?

[A] $(3x + 2)(2x + 3)$
[B] $(3x - 2)(2x + 3)$
[C] $(3x + 2)(2x - 3)$
[D] $(3x - 2)(2x - 3)$

When our pattern is applied to the polynomial in [C], we note that the middle term is obtained as follows:

$$(3x + 2)(2x - 3)$$
$$+$$
$$-9x + 4x$$
$$-5x$$

Since $-9x + 4x = -5x$, [C] is the correct choice.

Write the following indicated products as indicated sums.

16	$(2x + 3)(4x + 5)$ ______	$8x^2 + 22x + 15$
17	$(2x + 3)(4x - 5)$ ______	$8x^2 + 2x - 15$
18	$(3x - 2)(5x + 1)$ ______	$15x^2 - 7x - 2$
19	$(6x - 5)(2x - 1)$ ______	$12x^2 - 16x + 5$
20	$(x + 4)(x + 5)$ ______	$x^2 + 9x + 20$
21	$(2x + 3)(2x - 3)$ ______	$4x^2 - 9$
22	$(x + 6)(x + 6)$ ______	$x^2 + 12x + 36$

Did you notice that Items 21 and 22 each could have been done in another way? Item 21 involves the product of the sum and the difference of $2x$ and 3. In Item 22 you have $(x + 6)^2$.

Write $(-x + 2)(-x - 3)$ as an indicated sum.

23 $(-x + 2)(-x - 3) =$ ______ $x^2 + x - 6$

We might have noted that

24 $-x + 2 = \underline{-(\quad\quad)}$ — $-(x - 2)$

25 $-x - 3 = \underline{-(\quad\quad)}$ — $-(x + 3)$

Then $x^2 + x - 6 = (-x + 2)(-x - 3) = [-(x - 2)][-(x + 3)]$

26 $= (x - 2)(\underline{\quad\quad})$ — $x + 3$

We see from Items 24 and 25, that

$$x^2 + x - 6 = (-x + 2)(-x - 3)$$

$$= (x - 2)(x + 3)$$

If we are asked to factor $x^2 + x - 6$, we prefer the factorization $(x - 2)(x + 3)$.

We have [illegible]. Now we are ready to ask: Can we reason back from the sum to the product? Let us consider, for example, the polynomial $x^2 + [illegible]$.

We might ask: Are there integers r, s, m, n such that $(rx + m)(sx + n) = x^2 + [illegible]x + 12$?

Recall that Step 1 in writing $(rx + m)(sx + n)$ as an indicated sum is:

Step 1

$(rx + m)(sx + n)$

27 $\underline{\quad\quad}$ — rsx^2

If we want to find integers r, s, m, n such that $(rx + m)(sx + n) = x^2 + [illegible]x + 12$, we see that r

28 and s both must be 1 or both must be $\underline{\quad\quad}$. — -1

Our experience in Items 23 to 26 suggests that we will obtain the preferable factorization (if we can factor at all) by trying $r = s = \underline{\quad\quad}$. — 1

29

Thus if we wish to factor $x^2 + [illegible]x + 12$ over the integers, we need only look for integers m and n such that:

$$(x + m)(x + n) = x^2 + [illegible]x + 12.$$

Let us recall the steps for computing $(x + m)(x + n)$:

Step 1	Step 2	Step 3
$(x + m)(x + n)$	$(x + m)(x + n)$	$(x + m)(x + n)$
x^2	$(m + n)x$	mn
x^2 +	$8x$ +	12

	If $(x + m)(x + n) = x^2 + 8x + 12$, it must be true that:	
30	$m + n =$ ____	8
31	$mn =$ ____	12
	Moreover, m and n are integers.	
	What two integers have the product 12 and the sum 8?	
32	____ and ____. (You may recall answering similar questions in Chapter 12.)	6, 2 or 2, 6
33	We can check: $(x + 6)(x + 2) = x^2$ ____ x + ____.	$x^2 + 8x + 12$

	How did you find the answer in Item 32? One way is to list systematically all the pairs of integer factors of 12.	
	$12 = 1 \times 12$	
34	$= 2 \times$ ____	2×6
35	$=$ ____ $\times$ ____	3×4
	Of the pairs of factors listed, the pair whose sum is 8 is ____, ____.	
36		2, 6
37	$2 + 6 =$ ____, but $1 + 12$ ____ $(=, \neq)$ 8 and $3 + 4$ ____ 8.	8, $\neq$, $\neq$

The basic idea is simply to test pairs of factors until you find the right pair. This method of factoring is sometimes called <u>factoring by inspection</u>.

	If we wish to factor $x^2 + 5x + 6$ by inspection,	
38	we can notice that the coefficient of x^2 is ____.	1

Thus, proceeding as in the last example, we look for integers m and n such that

$$(x + m)(x + n) = x^2 + 5x + 6.$$

Thinking of our pattern, we thus need to find m and

39 n such that $mn =$ _____, 6

40 $m + n =$ _____. 5

41 We find that the pair of integers _____, _____ 2, 3

has the product 6 and the sum 5.

42 We check: $(x + 2)(x + 3) =$ _____ $x^2 + 5x + 6$

---

43 Factor $t^2 + 12t + 20$. _____ $(t + 2)(t + 10)$

(Notice that t^2 has the coefficient 1.)

If you had trouble, complete Items 44 to 48. If not, go to Item 49.

The factors of 20 are: 1, 20

44 2, _____ 2, 10

45 _____, _____ 4, 5

46 For which of these pairs is the sum 12? _____, _____ 2, 10

47 $t^2 + 12t + 20 =$ _____ $(t + 2)(t + 10)$

Factor over the integers:

48 $t^2 + 21t + 20$ _____ $(t + 20)(t + 1)$

49 $t^2 + 3t + 2$ _____ $(t + 2)(t + 1)$

50 $a^2 + 8a + 15$ _____ $(a + 5)(a + 3)$

To factor $x^2 + 2x - 15$, noting that the coefficient

51 of x^2 is _____, you might consider 1

$$(x + m)(x + n) = x^2 + 2x - 15.$$

52 We know from our pattern that $mn =$ _____. -15

53 Hence one of the integers m, n, must be _____ and the other positive. negative

	In order to factor $x^2 + 2x - 15$ over the integers,	
54	we need to find two integers whose product is _______	-15
	and whose sum is 2.	
	The pairs of integers whose product is -15 are:	
55	-1, 15 1, _____	-15
56	-3, 5 _____, _____.	5; 3, -5
	(Notice that we must consider -1, 15 and 1, -15 as different possibilities. Likewise we note both -3, 5 and 3, -5.)	
57	Of these pairs, the sum of _____ and _____ is 2.	5, -3
58	$x^2 + 2x - 15 =$ _______	$(x + 5)(x - 3)$
	Factor $a^2 - 3a - 18$.	
	The pair of factors whose product is -18 and whose	
59	sum is -3 is _______ and _______.	3 and -6
60	The factorization of $a^2 - 3a - 18$ is (____)(____).	$(a + 3)(a - 6)$
	If you were correct, skip to Item 67. If you had trouble, go ahead with the next items.	
	To factor $a^2 - 3a - 18$ we notice that we need two	
61	numbers whose product is _______ and whose sum is -3.	-18
	The pairs of integers with product -18 are:	
62	-1, 18 _____ _____	1, -18
63	-2, 9 _____, -9	2, -9
64	_______ _______	-3, 6; -6, 3
65	The pair whose sum is -3 is _____, _____.	-6, 3
66	Thus, $a^2 - 3a - 18 =$ (____)(____).	$(a + 3)(a - 6)$
	Consider $(x + m)(x + n) = x^2 - 8x + 15$.	
	Since $m \cdot n = 15$, m and n are either both positive	
67	or else both _______.	**negative**
68	Which pair of factors of 15 has the sum -8? _______	-3, -5

69	$x^2 - 8x + 15 = x^2 + (-3 - 5)x + 15$ $= (\_\_\_\_)(\_\_\_\_)$	$(x - 3)(x - 5)$

	Factor:	
70	$x^2 - 5x + 6$ ______	$(x - 2)(x - 3)$
71	$x^2 - 7x + 12$ ______	$(x - 3)(x - 4)$
72	$a^2 - 2a - 15$ ______	$(a + 3)(a - 5)$
73	$y^2 - 17y - 18$ ______	$(y + 1)(y - 18)$
74	$x^2 + 4x - 5$ ______	$(x + 5)(x - 1)$

In Item 74 we found:

$$x^2 + 4x - 5 = (x + 5)(x - 1)$$

by finding two integers whose product is -5 and whose sum is 4. We were factoring by inspection.

Recall that in Section 16-5 we used another method for factoring a polynomial such as $x^2 + 4x - 5$. This method depends on the idea of completing the square.

75	Factor $x^2 + 4x - 5$ by the method of completing the square. Then complete Item 75. $x^2 + 4x - 5$ can be written as the difference of the squares of ______ and ______. If you had trouble with Item 75, look back at Items [illegible] to 13, Sec. 16-5.	$(x + 2)$, 3

We thus have at our disposal two methods for factoring over the integers a polynomial of the form $x^2 + px + q$. (Needless to say, both lead to the same result.)

The trial and error method is quick, particularly where q has few pairs of factors. Later we will discuss some shortcuts that are helpful when q has many factors.

On the other hand, the method of completing the square is straightforward and direct, so you may prefer it in some cases. (This method is also important because it has other uses.)

Of course you must remember that not all polynomials over the integers can be factored over the integers. If one method fails to lead to a factorization, so will the other one.

	Let us determine whether $x^2 + 14x + 36$ is factorable over the integers.	
	List all of the pairs of factors of 36. There are	
76	______ different pairs. (how many)	five
77	Did you find a pair of factors whose sum is 14? ______	no
	Since we are unable to find such a pair of factors, we	
78	conclude that $x^2 + 14x + 36$ ______ (is, is not) factorable over the integers.	is not

	In attempting to factor $x^2 + 14x + 36$ we might instead have used the method of completing the square.	
79	$x^2 + 14x +$ ______ is a perfect square.	49
	$x^2 + 14x + 36 = x^2 + 14x + 49 - 49 + 36$	
80	$= (\quad\quad)^2$ ______	$(x + 7)^2 - 13$
81	13 ______ (is, is not) the square of an integer.	is not
	We cannot use the idea of difference of squares to factor $(x + 7)^2 - 13$ over the integers.	

	Factor over the integers if possible, using any method you prefer. Write "not factorable" if the polynomial cannot be factored over the integers.	
82	$t^2 + 9t + 20$ ______	$(t + 4)(t + 5)$
83	$t^2 + 10t + 20$ ______	not factorable
84	$x^2 + 20x + 64$ ______	$(x + 4)(x + 16)$
85	$x^2 + 16x + 64$ ______	$(x + 8)(x + 8)$ or $(x + 8)^2$
86	$x^2 + 8x + 64$ ______	not factorable
87	$x^2 - 9x + 8$ ______	$(x - 1)(x - 8)$

Factor $2x^2 + 3x + 1$ over the integers.

88 $2x^2 + 3x + 1 =$ ______ — $(2x + 1)(x + 1)$

If you had trouble with Item 88, complete Items 89 to 92. If not, go to Item 93.

89 In $2x^2 + 3x + 1$ the coefficient of x^2 is ____. — 2

We want to find integers r, s, m, n such that $(rx + m)(sx + n) = 2x^2 + 3x + 1$.

90 If this is to be true, rs must be ____ and mn — 2

91 must be ____. — 1

92 $(2x + 1)(x + 1) =$ ______ — $2x^2 + 3x + 1$

Hence Item 88, when completed, should read: $2x^2 + 3x + 1 = (2x + 1)(x + 1)$.

Some of the ideas about prime factorizations of integers, which were developed in Chapter 12, can help us find shortcuts in factoring polynomials over the integers.

For example, suppose we wished to factor

$$x^2 + 22x + 72.$$

The prime factorization of 72 is $2^3 \cdot 3^2$.

We want to have:

$$(x + m)(x + n) = x^2 + 22x + 72.$$

That is:

$$\begin{array}{cccc} x^2 & + (m + n)x & + & mn \\ \updownarrow & \updownarrow & & \updownarrow \\ x^2 & + \quad 22 \quad x & + & 72 \end{array}$$

Since $72 = 2^3 \cdot 3^2$, we see that the only prime

93 factors of m and n are 2 and ____. — 3

94 At least one of the numbers m and n is $\underset{\text{(odd, even)}}{\underline{\hspace{5em}}}$. — even

95 22 is also $\underset{\text{(odd, even)}}{\underline{\hspace{5em}}}$. — even

Since one of the numbers m and n is even, and since their sum, 22, is even, it is evident that <u>both</u> the required factors must be even. On the other hand, since 3 is not a factor of 22, we cannot have 3 as a factor of both m and n.

Thus we have excluded, from the list of pairs of factors of 72, all those except

96 _____, _____ and _____, _____. — 36, 2 and 4, 18

You can then conclude easily:

97 $x^2 + 22x + 72 = (\quad)(\quad)$. — $(x + 4)(x + 18)$

If you had trouble with Item 96, write a complete list of pairs of factors of 72. Then review how each pair may be excluded by the argument above except those in Item 96.

Example: The pair 9, 8 is excluded because 9 is

98 ________ (even, odd) and hence 9 + 8 is odd. — odd

Let us demonstrate this line of reasoning again by considering

$$x^2 + 55x + 600.$$

We could begin by listing all the pairs of integers

*99 whose _______ is 600. — product

This list, however, would be quite long. Let us instead begin by considering the prime factorization of 600.

*100 $600 = 2^3 \cdot \_\_\_ \cdot \_\_\_$ — $2^3 \cdot 3 \cdot 5^2$

Thus, if $(x + m)(x + n) = x^2 + 55x + 600$, m and n

*101 will contain only 2's, 3's and ___'s as factors. — 5's

The fact that m + n, which equals 55, is odd

*102 means that m and n ________ (can, cannot) both be even or both be odd. — cannot

Therefore, the three 2's must all be in the same factor.

	Further, since 55 is divisible by 5, and since we	
*103	have ______ (how many) 5's to split between m and n, it	two
	is evident that 5 must be a factor of both m and n.	
	We have decided that either m or n must have 2^3 as a factor. Moreover, m and n each have 5 as a factor.	
	This leaves a factor of 3 for either m or n.	
	This gives us only two possibilities for m and n:	
	either $m = 2^3 \cdot 5 \cdot 3$ and $n = 5$	
*104	or $m = 2^3 \cdot 5$ and $n =$ ______.	$5 \cdot 3$

*105	In either case $mn =$ ______.	600
	Since $m + n$ must equal 55 we chose the pair	
*106	______, ______.	40, 15
*107	Thus, $x^2 + 55x + 600 =$ ______.	$(x + 40)(x + 15)$

	Factor:	
*108	$a^2 - 21a + 108$	$(a - 12)(a - 9)$
*109	$a^2 + 25a - 600$	$(a + 40)(a - 15)$

17-2. Factoring by Inspection, Continued

	As we have observed:	
1	$(rx + m)(sx + n) =$ ______ $x^2 + (rn + ms)x + mn$	rsx^2
	If we wish to factor a quadratic polynomial over the integers we may work with this pattern.	
	In the last section, we factored such polynomials as $x^2 + 21x + 20$, $x^2 - 5x + 6$, $x^2 - 7x + 12$. These polynomials have the general form $x^2 + px + q$.	
2	The coefficient of x^2 in each of these cases is ______.	1

A quadratic polynomial having -1 as the coefficient of x^2, offers no difficulty.

Item		Answer
	Consider $-x^2 - x + 12$.	
	If we wish to factor this polynomial, it is helpful to write:	
3	$-x^2 - x + 12 = \underline{-(\qquad)}$	$-(x^2 + x - 12)$
4	$x^2 + x - 12 = \underline{(\qquad)(\qquad)}$	$(x+4)(x-3)$
	Hence $-x^2 - x + 12 = -(x^2 + x - 12)$	
5	$= \underline{-(\qquad)(\qquad)}$	$-(x+4)(x-3)$

Item		Answer
	Factor if possible:	
6	$-x^2 - 13x - 12$ ______	$-(x+12)(x+1)$
7	$12 - 11x - x^2$ ______	$-(x+12)(x-1)$
8	$-x^2 + 5x + 12$ ______	not factorable
9	$x^2 - 7x + 12$ ______	$(x-3)(x-4)$
10	$t^2 + 3t + 2$ ______	$(t+2)(t+1)$
11	$2t^2 + 6t + 4$ ______	$2(t+2)(t+1)$

Item		Answer
	In Item 11, you may have noticed that	
12	$2t^2 + 6t + 4 = \underline{2(t^2 + \qquad)}$.	$2(t^2 + 3t + 2)$
	If you did, then you saw that the complete factorization could be found by factoring $t^2 + 3t + 2$, which you did in Item 10. Thus	
	$2t^2 + 6t + 4 = 2(t^2 + 3t + 2)$	
13	$= \underline{2(\qquad)(\qquad)}$	$2(t+2)(t+1)$
	If you did not proceed this way, you could have found by trial and error that	
14	$2t^2 + 6t + 4 = (2t + 2)(\underline{\qquad})$	$t + 2$
15	$= \underline{2(\qquad)(t + 2)}$	$2(t+1)(t+2)$
	In completing Item 14 you used the fact that the	
16	terms in $2t + 2$ have the common factor _____.	2

You might also have found the complete factorization of $2t^2 + 6t + 4$ in these steps:

$$2t^2 + 6t + 4 = (t + 1)(2t + 4)$$

17 $= 2(\quad)(\quad)$ — $2(t+1)(t+2)$

Comparing Items 13, 15, and 17, we see (of course!) the same complete factorization. We also see--this is the important point--that in factoring $2t^2 + 6t + 4$ the simplest method is to recognize first the common factor 2.

The preceding example illustrates two important ideas:

1) In factoring an indicated sum, it is wise to begin by seeing whether there is a common factor in each of the terms.

2) If $ax^2 + bx + c = (rx + m)(sx + n)$, and if the terms in $rx + m$ have a common factor, then so do the terms in $ax^2 + bx + c$.

Factor completely: $7x^2 + 14x + 7$.

18 $7x^2 + 14x + 7 =$ ______ — $7(x + 1)^2$

Try this one: $2x^2 + 7x + 3$.

19 In this case the terms ______ (do, do not) have a common factor. — do not

By trying various possibilities, we find:

20 $2x^2 + 7x + 3 = (\quad)(\quad)$. — $(2x+1)(x+3)$

If you completed the last item correctly, you observed that it was necessary to find integers r, s, m, n such that

$$(rx + m)(sx + n) = 2x^2 + 7x + 3.$$

You may have observed that when you write

21 $(rx + m)(sx + n)$ as a sum the first term is ____ x^2. — rsx^2

22 Thus you could conclude: $rs =$ ____. The factors you are seeking have the form: — 2

$$(2x + m)(x + n).$$

23 Also you could have noted: mn must be ____. — 3

You thus may have simply tried out

$$(2x + 1)(x + 3)$$
$$(2x + 3)(x + 1)$$

since these are the only possibilities.

From which of these products do you obtain $2x^2 + 7x + 3$?

24 ____ — $(2x + 1)(x + 3)$

25 Which of the following sentences is correct?

[A] $2x^2 + 5x + 3 = (2x + 3)(x + 1)$

[B] $2x^2 + 5x + 3 = (2x - 3)(x - 1)$

[C] $2x^2 + 5x + 3 = (2x + 1)(x + 3)$

The correct response is [A].

If possible, factor. If not, write "not factorable".

26 $3a^2 + 4a - 7$ ____ — $(3a + 7)(a - 1)$

27 $3a^2 - 4a - 7$ ____ — $(3a - 7)(a + 1)$

28 $-3a^2 - 4a + 7$ ____ — $-(3a + 7)(a - 1)$

29 $2x^2 + 9x + 3$ ____ — not factorable

Note that in the polynomial $2x^2 + 9x + 3$, the maximum sum of the inside and outside products that can be obtained is 7, which is less than 9.

Let us try another example. Suppose we wish to factor $6x^2 + 19x + 10$. We want, then, to find integers r, s, m, n such that

$$(rx + m)(sx + n) = 6x^2 + 19x + 10.$$

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We can compare our polynomial with the pattern:

$$\begin{array}{c} (rx + m)(sx + n) = \underset{\updownarrow}{rsx^2} + \underset{\updownarrow}{(\underbrace{rn + ms})}x + \underset{\updownarrow}{mn} \\ \qquad\qquad\qquad 6x^2 + \quad 19 \quad x + 10 \end{array}$$

30	We note that $rs =$ ____ and $mn =$ ____.	6, 10
31	The factorizations of 6 are $6 \cdot$ ___ and ___ $\cdot$ ___.	$6 \cdot 1$, $2 \cdot 3$
32	The factorizations of 10 are ___ $\cdot 1$ and ___ $\cdot$ ___.	$10 \cdot 1$, $2 \cdot 5$

We can list all the possibilities we need to test:

1) $(x + 1)(6x + 10)$
2) $(x + 2)(6x + 5)$
3) $(x + 5)(6x + 2)$
4) $(x + 10)(6x + 1)$
5) $(2x + 1)(3x + 10)$
6) $(2x + 2)(3x + 5)$
7) $(2x + 5)(3x + 2)$
8) $(2x + 10)(3x + 1)$

Note that in each case $rs = 6$ and $mn = 10$. Therefore, we need only select the pair of factors which gives us $(rn + ms) = 19$. We determine that this is true only for case (7).

Trying out each case is tedious. Can we eliminate any cases without actually testing?

33	In 8), we observe that the first factor, $(2x + 10)$, can be written as ___ $(x + 5)$. Thus 8) is not a possibility, because if it were, 2 would be a factor of each term of the original polynomial.	$2(x + 5)$
34	Likewise in 6) we see that the first factor, $2x + 2$, can be written as ______. Using the same reasoning we can eliminate this case as a possibility.	$2(x + 1)$
35	Using the same reasoning, we can also eliminate two others; namely ____ and ____.	1), 3)
	Having ruled out some entries in our list, we can test the rest.	

36	We find: $6x^2 + 19x + 10 =$ ______.	$(2x+5)(3x+2)$

Factoring by inspection is really only a matter of trial and error. You do not need to list or try out all the possibilities. Stop when you have found a product which does yield the given sum.

	Let's try another: Factor $3x^2 - 2x - 21$.	
37	The terms ______ (do, do not) have a common factor.	do not
	We must find integers r, s, m, n such that $$(rx + m)(sx + n) = 3x^2 - 2x - 21.$$	
38	The constant in our polynomial is ____. Hence one of the numbers m and n must be positive and the	-21
39	other ______.	negative
40	We can factor 3 in only one way: $3 =$ ____.	$3\cdot1$ or $1\cdot3$
41	We can factor 21 in two ways: ______ and ______.	$21\cdot1$ and $7\cdot3$
	Recall that we do not need to test $(3x - 3)(x + 7)$, because the terms in $3x - 3$ have a common factor;	
42	namely ____.	3
43	$3x^2 - 2x - 21 = (\quad)(\quad)$	$(x-3)(3x+7)$

	Factor:	
44	$4y^2 + 23y - 6 =$ ______	$(4y-1)(y+6)$
45	$x^2 + 4x - 32 =$ ______	$(x+8)(x-4)$
46	$8a^2 + 10a - 3 =$ ______	$(4a-1)(2a+3)$

Here is an (optional) example in which the prime factorizations of the coefficients are helpful. You may wish to try completing Item *54 at once, or you may prefer to go directly through Items *47 to *54. (If you wish to omit the starred items, go to Item 56.)

	Factor $25x^2 - 45x - 36$.	
*47	We have as prime factorizations: $25 =$ ______	5^2
*48	and $36 =$ ______.	$2^2 \cdot 3^2$
*49	The sum of the inside and outside products is ______.	-45
*50	Since 45 is a multiple of 5, and since we have two 5's to put somewhere, both the inside and the outside products will contain ______ as a factor.	5
*51	This suggests trying: $25x^2 - 45x - 36 = (\_\_x + \text{something})(\_\_x + \text{something})$.	5, 5
	Since 45 is also a multiple of 3, we also expect to find that 3, which occurs twice as a factor of 36, is a factor of both the inside and the outside products.	
*52	45 is ______ (odd, even), which tells us that the 2's	odd
*53	______ (can, cannot) be in both inside and outside products.	cannot
	We have thus eliminated all possibilities except.	
	$(5x + 12)(5x - 3)$	
	$(5x - 12)(5x + 3)$	
*54	We find: $25x^2 - 45x - 36 = (\quad)(\quad)$.	$(5x-12)(5x+3)$

	Factor:	
*55	$6x^2 + 7x - 24 =$ ______	$(2x-3)(3x+8)$

*56 Can the quadratic polynomial $2x^2 + ax + b$ be factored if a is even and b is odd?

[A] yes [B] no

There is only one factor 2 in the coefficient of x^2 and none in the constant term. Therefore, either the inside product or the outside product will have a factor of 2 but not both. Thus, the sum of the inside and outside products will be odd. The answer is [B].

Although this section has emphasized factoring by inspection, you should not forget that some polynomials can be recognized as perfect squares and others as the difference of two squares. In the following list of examples you will find occasions to use all the methods we have studied.

57	$9x^2 + 12x + 4 =$ ______	$(3x + 2)^2$
58	$9a^2 + 3a - 2 =$ ______	$(3a+2)(3a-1)$
59	$9a^2 + 3a =$ ______	$3a(3a+1)$
60	$9a^2 + 9 =$ ______	$9(a^2 + 1)$
61	$12x^2 - 51x + 45 =$ ______	$3(x-3)(4x-5)$

62	$10x^2 + 43x + 45 =$ ______	$(5x+9)(2x+5)$
63	$10x^2 - 69x - 45 =$ ______	$(2x-15)(5x+3)$
64	$6 - 23a - 4a^2 =$ ______	$-(4a-1)(a+6)$
65	$6 - 3x^2 + 17x =$ ______ (Caution)	$-(3x+1)(x-6)$
66	$19x - 6 + 7x^2 =$ ______	$(7x-2)(x+3)$
67	$p^2 + 2pq + q^2 =$ ______	$(p + q)^2$
68	$25x^2 - 70xy + 49y^2 =$ ______	$(5x - 7y)^2$
69	$2a^4 + 20a^3 + 50a^2 =$ ______	$2a^2(a + 5)^2$
70	$2a^2 + 15a + 25 =$ ______	$(2a+5)(a+5)$
71	$w^2 - 16 =$ ______	$(w - 4)(w + 4)$
72	$(x + 3)^2 - 16 =$ ______	$(x - 1)(x + 7)$

In much of this section we have considered quadratic polynomials. However, the ideas we have developed can be applied in certain other cases.

$z^6 - 5z^3 - 14$ is not a quadratic polynomial. However, if we write

$$z^6 - 5z^3 - 14 = (z^3)^2 - 5(z^3) - 14$$

we are able to factor the expression.

73 Thus, $z^6 - 5z^3 - 14 = (\qquad)(\qquad)$. $(z^3+2)(z^3-7)$

Factor completely:

74 $a^4 - 13a^2 + 36$ ______ [Hint: Your final result should contain four factors.] $(a+2)(a-2)(a+3)(a-3)$

75 $b^4 - 11b^2 + 28$ ______ $(b^2-7)(b+2)(b-2)$

76 $y^4 - 81$ ______ $(y^2+9)(y+3)(y-3)$

We have been factoring polynomials over the integers. Now let us consider the polynomial

$$\frac{x^2}{4} + \frac{5x}{4} + \frac{3}{2} .$$

This is not a polynomial over the integers. However, we can factor this polynomial, using familiar ideas.

We may write $\frac{1}{4}x^2 + \frac{5}{4}x + \frac{3}{2}$ as the indicated product of a rational number and a polynomial over the integers. Thus,

$$\frac{1}{4}x^2 + \frac{5}{4}x + \frac{3}{2} = \frac{1}{4}x^2 + \frac{5}{4}x + \frac{6}{4}$$

77 $= \frac{1}{4}(\qquad)$ $\frac{1}{4}(x^2 + 5x + 6)$

The advantage of this is that we know how to factor $x^2 + 5x + 6$.

78 $x^2 + 5x + 6 = (\qquad)(\qquad)$ $(x + 2)(x + 3)$

Therefore $\frac{1}{4}x^2 + \frac{5}{4}x + \frac{3}{2} = \frac{1}{4}(x^2 + 5x + 6)$

79 $=$ ______ $\frac{1}{4}(x+2)(x+3)$

Here is another example:

$$\frac{1}{3}x^2 + \frac{2}{3}x + \frac{1}{4} = \frac{4}{12}x^2 + \frac{8}{12}x + \frac{3}{12}$$

80 $= \frac{1}{12}(\qquad + 3)$ $\frac{1}{12}(4x^2 + 8x + 3)$

81 $= \frac{1}{12}(\qquad)(\qquad)$ $\frac{1}{12}(2x+3)(2x+1)$

Notice in Item 80, that 12 is the least common

82 denominator of the ______ of the polynomial. **coefficients**

Here, as before, we first wrote the polynomial as the product of a rational number and a polynomial over the integers; then we factored the polynomial over the integers.

Let us examine one more example. Consider

$$\frac{x^2}{6} + \frac{x}{2} + \frac{1}{6}.$$

[illegible]

83 $$\frac{x^2}{6} + \frac{x}{2} + \frac{1}{6} = \underline{\quad}(\underline{\qquad\qquad}).$$ $\frac{1}{6}(x^2 + 3x + 1)$

84 Unfortunately, $x^2 + 3x + 1$ ________ (can, cannot) be factored over the integers. **cannot.**

When you try to factor $(x^2 + 3x + 1)$ over the integers you find that there are no integers m and n such that

$$(x + m)(x + n) = x^2 + 3x + 1.$$

Notice that if there were such integers m and n, then mn would be 1. Hence the only possibilities you need to test are $(x + 1)(x + 1)$ and $(x - 1)(x - 1)$. From neither product do you obtain $3x$ for the middle term.

You might wonder whether you can find rational numbers m and n for which

$$(x + m)(x + n) = x^2 + 3x + 1.$$

This might lead you to try out such products as $(x + \frac{1}{2})(x + 2)$, and $(x + \frac{3}{4})(x + \frac{4}{3})$, and the like. That is, you might expect more chance of factoring when you have more numbers to choose from. It turns out that you cannot find rational values for m and n. Polynomials which cannot be factored over the integers will be discussed in more detail in the next section.

Factor.

85 $\frac{1}{2}t^2 - 3t + 4 =$ ________ **$\frac{1}{2}(t - 2)(t - 4)$**

86 $\frac{1}{2}t^3 - 3t^2 + 4t =$ ________ **$\frac{1}{2}t(t - 2)(t - 4)$**

87 $\frac{1}{3}x^2 + \frac{4}{3}x + \frac{4}{3} =$ ________ **$\frac{1}{3}(x + 2)^2$**

	$\frac{3}{4}x - 1 = \frac{3}{4}x - \frac{4}{4}$	
88	$= \_\_\_\_\_\_$	$\frac{1}{4}(3x - 4)$
89	$\frac{2}{5}x^2 - 4 = \_\_\_\_\_\_$	$\frac{2}{5}(x^2 - 10)$
90	$\frac{1}{3}x^2 + \frac{1}{4}x + \frac{1}{6} = \_\_\_\_\_\_$	$\frac{1}{12}(4x^2+3x+2)$

Do not forget that factoring is useful in solving equations.

	Solve:	
91	$6y^2 + y - 1 = 0$ ______	$\{\frac{1}{3}, -\frac{1}{2}\}$
	If you had trouble with Item 91, complete Items 92 to 94. If not, go to Item 95.	
92	$6y^2 + y - 1 = \underline{(3y \quad)(2y \quad)}$	$(3y - 1)(2y + 1)$
	Hence the open sentence $6y^2 + y - 1 = 0$ is equivalent to the sentence	
93	$3y - 1 = 0$ or ______.	$2y + 1 = 0$
94	The solution set of this compound sentence is ______.	$\{\frac{1}{3}, -\frac{1}{2}\}$

	Solve:	
95	$8x^2 + 10x - 3 = 0$ ______	$\{\frac{1}{4}, -\frac{3}{2}\}$
96	$9x^2 = 4x$ ______ [Hint: Begin by writing $9x^2 - 4x = 0$.]	$\{0, \frac{4}{9}\}$
97	$y^2 - 13y + 36 = 0$ ______	$\{4, 9\}$

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17-3. Factoring over the Real Numbers

Thus far in our work on factoring we have emphasized factoring polynomials over the integers.

	In the polynomial $x^2 - 6x + 8$, the coefficients	
1	are ____, ____, ____.	1; -6, 8
	The set of coefficients, $\{1, -6, 8\}$, is a subset of the set of integers. Hence we have called $x^2 - 6x + 8$ a polynomial over the integers.	
2	The degree of the polynomial is ____; that is, it is a quadratic polynomial.	2
3	We see: $x^2 - 6x + 8 = ($____$)($____$)$.	$(x - 2)(x - 4)$
	In completing Item 3, you factored $x^2 - 6x + 8$ over the integers. That is, you found a factorization	
4	using only polynomials over the _____.	integers
	In Item 3, the quadratic polynomial is written as	
5	a product of two polynomials, each having degree ____.	1

In this section we are not going to restrict ourselves to polynomials over the integers, but most of our discussion will still relate to quadratic polynomials.

6 Which of the following are quadratic polynomials?

P. $3x - 5$ S. $x^2 + 5x - \frac{4}{[illegible]}$

Q. $\sqrt{2}\,x^2 - 5$ T. $x^2 - 2$

R. $x^2 + |x| + 1$

[A] P and T

[B] P, R, S, and T

[C] Q, S, and T

[D] P, Q, S, and T

$3x - 5$ is a polynomial, but its degree is 1 so it is <u>not</u> a quadratic polynomial. $x^2 + |x| + 1$ is <u>not</u> a polynomial, since it involves $|x|$. You should have chosen [C].

We are going to be interested in factoring quadratic polynomials, but we will not restrict ourselves to factorizations over the integers. To emphasize our present point of view, we will sometimes speak of factoring a polynomial over the real numbers.

	We can write $2x^2 + 3x$ as a product of polynomials in many ways.	
7	$2x^2 + 3x = x(\_\_\_\_\_)$	$x(2x + 3)$
8	$= 2(\_\_\_\_\_)$	$2(x^2 + \frac{3}{2}x)$
9	$= 2x(\_\_\_\_\_)$	$2x(x + \frac{3}{2})$
10	$= \frac{x}{4}(\_\_\_\_\_)$	$\frac{x}{4}(8x + 12)$
	In each of Items 7 to 10 we have written the quadratic polynomial $2x^2 + 3x$ as a product of two polynomials.	

We will be most interested in writing a quadratic polynomial as the product of two polynomials of degree 1. Items 7, 9, and 10 illustrate such factorizations. In Item 8 one factor is of degree 2. We would not regard it as a complete factorization of $2x^2 + 3x$. Item 7 is the simplest factorization, but we will find that at times we have special reasons for using factorizations like that in Item 9.

In factoring quadratic polynomials we will make use of what we have learned about perfect squares and about factoring the difference of two squares.

	Recall again that the sentences	
11	$a^2 - b^2 = \_\_\_\_\_$	$(a + b)(a - b)$
12	$a^2 + 2ab + b^2 = \_\_\_\_\_$	$(a + b)^2$
13	$a^2 - 2ab + b^2 = \_\_\_\_\_$	$(a - b)^2$
	are true for all real values of a and b.	

In factoring polynomials over the real numbers we can apply these patterns.

Consider the polynomial $x^2 - 2$. This is the

14 ______________ of x^2 and 2. — **difference**
(sum, difference)

15, 16 x^2 is the square of _____. However, 2 ____________ (is, is not) the square of an integer. — **x, is not**

Hence if we are factoring over the integers we must regard $x^2 - 2$ as not factorable.

17 On the other hand, $2 = (\quad)^2$. — $(\sqrt{2})^2$

18 2 ____________ (is, is not) the square of a real number. — **is**

If we are factoring over the real numbers, we can

19 regard $x^2 - 2$ as the ______ of two squares. — **difference**

That is, $x^2 - 2 = x^2 - (\sqrt{2})^2$.

20 We may thus write: $x^2 - 2 = (\quad)(\quad)$. — $(x+\sqrt{2})(x-\sqrt{2})$

We notice that $x + \sqrt{2}$ and $x - \sqrt{2}$ are polynomials of first degree.

21 Which of the following is not the square of a real number?

[A] $\frac{9}{4}$ [C] $\sqrt{3}$

[B] 11 [D] -4

$(\frac{3}{2})^2 = \frac{9}{4}$

$(\sqrt{11})^2 = 11$

$\sqrt{3}$ is a non-negative real number, approximately 1.732.

$\sqrt{\sqrt{3}}$ is consequently a non-negative real number, approximately 1.32.

-4 is negative and hence is not the square of a real number.

You should have chosen [D].

Any non-negative real number is the square of a real number. On the other hand, no negative real number is the square of a real number.

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22	Which of the following is <u>not</u> factorable over the real numbers? [A] $x^2 - 5$ [C] $x^2 + 5$ [B] $5x^2 - 7$ [D] $7 - 5x^2$

$x^2 - 5 = (x + \sqrt{5})(x - \sqrt{5})$;

$5x^2 - 7 = (\sqrt{5}x + \sqrt{7})(\sqrt{5}x - \sqrt{7})$;

$7 - 5x^2 = (\sqrt{7} + \sqrt{5}x)(\sqrt{7} - \sqrt{5}x)$.

$x^2 + 5$ is not factorable over the real numbers, so [C] is the correct choice. Notice that $x^2 + 5$ is the <u>sum</u> of x^2 and $(\sqrt{5})^2$.

It is clear how we may factor a polynomial such as $x^2 - c$ $(c \geq 0)$ over the real numbers:

$$x^2 - c = (x + \sqrt{c})(x - \sqrt{c}), \quad (c \geq 0).$$

Factor each of the following over the real numbers, if possible.

23	$y^2 - 23 =$ ______	$(y+\sqrt{23})(y-\sqrt{23})$
24	$4x^2 - \frac{9}{25} =$ ______	$(2x+\frac{3}{5})(2x-\frac{3}{5})$
25	$3w^2 - 4 =$ ______	$(\sqrt{3}w+2)(\sqrt{3}w-2)$
26	$4x^2 + 1 =$ ______	not factorable

Examine:

$$x^2 + 2\sqrt{3}x + 3$$

This phrase is a polynomial.

$$x^2 + 2\sqrt{3}x + 3 = x^2 + 2\sqrt{3}x + (\sqrt{3})^2$$

Compare $x^2 + 2\sqrt{3}x + (\sqrt{3})^2$ with $a^2 + 2ab + b^2$:

$$\begin{array}{ccccccc} a^2 & + & 2a & b & + & b^2 \\ \downarrow & & \downarrow & \downarrow & & \downarrow \\ x^2 & + & 2x & \sqrt{3} & + & (\sqrt{3})^2 \end{array}$$

27	Since $a^2 + 2ab + b^2 = (\_\_\_\_\_)^2$,	$(a + b)^2$
28	we have $x^2 + 2\sqrt{3}x + 3 = (\_\_\_\_\_)^2$.	$(x + \sqrt{3})^2$
29	Similarly: $x^2 + 4\sqrt{5}x + 20 = (\_\_\_\_\_)^2$.	$(x + 2\sqrt{5})^2$

Suppose we wish to factor the polynomial $x^2 + 4x - 2$. If we were asked to factor this polynomial over the <u>integers</u> we might look for integers m, n such that

$$x^2 + 4x - 2 = (x + m)(x + n).$$

30	It turns out that we ______ (can, cannot) find such integers.	cannot
31	That is, $x^2 + 4x - 2$ ______ (is, is not) factorable <u>over the integers</u>.	is not

In trying to factor $x^2 + 4x - 2$ over the integers, we might have preferred to try completing the square. Do this. Then complete Item 33.

32	$x^2 + 4x - 2 = \underline{(x + 2)^2 - \quad\quad}$	$(x + 2)^2 - 6$
	There is no <u>integer</u> whose square is 6. Hence again we would conclude: $x^2 + 4x - 2$ is not factorable	
33	over the ______.	integers
	Suppose, however, that we wish to factor $x^2 + 4x - 2$ <u>over the real numbers</u>.	
34	Since $(\quad)^2 = 6$, we see that $x^2 + 4x - 2 = (x + 2)^2 - (\sqrt{6})^2$	$(\sqrt{6})^2$
35	$= (x + 2 + \sqrt{6})(\quad\quad)$	$(x + 2 - \sqrt{6})$
36	We have factored $x^2 + 4x - 2$ over the ______ <u>numbers</u>.	real numbers

	Suppose you are asked to factor $x^2 - 6x + 8$ over the real numbers.	
	You might still wish to check first whether you can factor $x^2 - 6x + 8$ over the integers.	
37	It is easy to see that $x^2 - 6x + 8 = (\quad)(\quad)$.	$(x - 4)(x - 2)$
	We have factored $x^2 - 6x + 8$ over the integers, but of course when we are factoring over the integers we are also factoring over the real numbers.	

If you are asked, then, to factor a quadratic polynomial over the real numbers you may begin, if you like, by trying to factor it by inspection.

However, if you fail in doing this then you will need to apply the method of completing the square.

	Let us try this method on $x^2 - 6x + 6$.	
38	$x^2 - 6x + 6 = (x^2 - 6x + \_\_\_\_) - \_\_\_\_ + 6$	9, 9
39	$= (x - 3)^2 - (\_\_\_\_)^2$	$(x-3)^2 - (\sqrt{3})^2$
40	$= (\_\_\_\_)(\_\_\_\_)$	$(x-3+\sqrt{3})(x-3-\sqrt{3})$

	Factor over the real numbers by completing the square:	
41	$x^2 - 4x - 1 = (\_\_\_\_)(\_\_\_\_)$	$(x-2+\sqrt{5})(x-2-\sqrt{5})$
42	$y^2 + 2y - 9 = (\_\_\_\_)(\_\_\_\_)$	$(y+1+\sqrt{10})(y+1-\sqrt{10})$
43	$z^2 - 8z + 16 = (\_\_\_\_)(\_\_\_\_)$	$(z-4)^2$ [Did we fool you?]
44	$a^2 - 6a + 3 = (\_\_\_\_)(\_\_\_\_)$	$(a-3+\sqrt{6})(a-3-\sqrt{6})$

You must not be led to believe that this technique of completing the square will enable us to factor every polynomial over the real numbers.

	Consider $x^2 - 4x + 6 = x^2 - 4x + 4 - 4 + 6$	
45	$= (x - \_\_\_\_)^2 + 2$	$(x - 2)^2 + 2$
46	This last is <u>not</u> the difference of two ______.	squares
	In fact, $x^2 - 4x + 6$ is not factorable over the real numbers.	

47	Recalling that $(a + b)^2 = a^2 + \_\_\_\_ + b^2$,	2ab.
	complete the following to form a true sentence:	
48	$x^2 + 3x + \_\_\_\_ = (x + \_\_\_\_)^2$	$\frac{9}{4}$; $(x + \frac{3}{2})^2$
49	$y^2 + y + \_\_\_\_ = (y + \_\_\_\_)^2$	$\frac{1}{4}$; $(y + \frac{1}{2})^2$

Factor by completing the square. Answers are on page xix.

50. $a^2 + 3a + 1$
51. $y^2 + y - 3$
52. $x^2 - 5x - 2$
53. $y^2 + \frac{2}{3}y - 1$

	Consider the polynomial $2x^2 + 8x + 3$.	
54	In this polynomial the coefficient of x^2 is _____.	2
	If we wish to factor this polynomial by completing the square, it is helpful to begin by writing:	
55	$2x^2 + 8x + 3 = 2(\underline{x^2 + \quad}) + 3$	$(x^2 + 4x)$
56	We recognize that $x^2 + 4x +$ _____ $= \underline{(x + \quad)^2}$.	4; $(x + 2)^2$
57	We also recall that $2(x^2 + 4x + 4) = 2x^2 + 8x +$ _____.	8
	Thus we have: $2x^2 + 8x + 3 = 2(x^2 + 4x) + 3$	
58	$= 2(x^2 + 4x + 4) - \underline{\quad\quad} + 3$	-8
59	$= \underline{2(x + 2)^2 - \quad}$	$2(x + 2)^2 - 5$
	In Item 58 you should have noticed that adding $\underline{4}$ inside the parentheses is the same as adding $\underline{8}$ to the polynomial.	

Let us summarize the steps in factoring $2x^2 + 8x + 3$ by completing the square.

$$\begin{aligned} 2x^2 + 8x + 3 &= 2(x^2 + 4x) + 3 \\ &= 2(x^2 + 4x + 4) - 8 + 3 \\ &= 2(x + 2)^2 - 5 \\ &= (\sqrt{2}(x + 2) + \sqrt{5})(\sqrt{2}(x + 2) - \sqrt{5}) \\ &= (\sqrt{2}x + 2\sqrt{2} + \sqrt{5})(\sqrt{2}x + 2\sqrt{2} - \sqrt{5}) \end{aligned}$$

Once again, we find that our knowledge about factoring can be applied when we wish to solve equations.

	Consider the equation $a^2 - 4a + 1 = 0$.	
	Completing the square, we write the equivalent equation.	
60	$\underline{(a - \quad)^2 - (\quad)^2 = 0}$	$(a-2)^2-(\sqrt{3})^2=0$
61	Thus we have: $(a - 2 - \sqrt{3})(a - 2 + \sqrt{3}) =$ _____.	0
	The equivalent compound sentence,	
62	$a - 2 - \sqrt{3} = 0$ or ______.	$a - 2 + \sqrt{3} = 0$
	This sentence, and hence the original equation, has	
63	the solution set ______.	$\{2+\sqrt{3}, 2-\sqrt{3}\}$

Solve:

64 $x^2 + 4x - 2 = 0$ ______ $\{-2-\sqrt{6}, -2+\sqrt{6}\}$

[Hint: If you have trouble, refer to Items 33 to 35.]

65 $a^2 - 4a + 15 = 0$ ______ $\emptyset$

Notice that in completing Item 65 you find the equivalent equation $(a - 2)^2 + 11 = 0$. Since 11 is positive and $(a - 2)^2$ is non-negative for all real values of a, their sum is greater than 0 for all real values of a.

In the course of this section we have found:

$$x^2 + 4x - 2 = (x + 2)^2 - 6 \qquad \text{(Item 32)}$$
$$x^2 - 6x + 6 = (x - 3)^2 - 3 \qquad \text{(Item 39)}$$
$$x^2 - 4x + 6 = (x - 2)^2 + 2 \qquad \text{(Item 45)}$$
$$x^2 - 5x - 2 = (x - \tfrac{5}{2})^2 - \tfrac{33}{4} \qquad \text{(Item 52)}$$
$$2x^2 + 8x + 3 = 2(x + 2)^2 - 5 \qquad \text{(Item 59)}$$

Notice that in each of these instances we began with a polynomial of the form $ax^2 + bx + c$. We were able to write this polynomial in the form

$$a(x - h)^2 + k.$$

Thus we observe the pattern:

$$\begin{array}{ccccccc} 2x^2 & + 8x & + 3 & = 2(x & + 2)^2 & + & (-5) \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow \\ ax^2 & + bx & + c & = a(x & - h)^2 & + & k \end{array}$$

66 We observe that k corresponds to ______ and that -5

67 h corresponds to ______. -2

By now you should realize that every quadratic polynomial can be written in the form

$$a(x - h)^2 + k.$$

This is sometimes called the <u>standard form</u> of a quadratic polynomial.

Notice that writing a quadratic polynomial in standard form is really only an application of completing the square.

Write each quadratic polynomial in standard form.

68	$x^2 + 4x + 8 = (\_\_\_\_)^2 + \_\_\_\_$	$(x + 2)^2 + 4$
69	$x^2 + 3x - 4 = \_\_\_\_$	$(x+\frac{3}{2})^2 - \frac{25}{4}$
70	$x^2 - x + 2 = \_\_\_\_$	$(x - \frac{1}{2})^2 + \frac{7}{4}$
71	$-x^2 - 2x + 3 = \_\_\_\_$	$-(x + 1)^2 + 4$

If you had trouble with Item 71, complete Items 72 and 73. If not, go on to Item 74.

72	$-x^2 - 2x + 3 = -(\_\_\_\_)$	$-(x^2 + 2x - 3)$
73	$= -((x + 1)^2 - \_\_\_\_)$	-4
	$= -(x + 1)^2 + 4$	

Write in standard form:

74	$3x^2 - 4x + 5 = \_\_\_\_$	$3(x-\frac{2}{3})^2 + \frac{11}{3}$

If you had trouble with Item 74, complete Items 75 and 76.

75	$3x^2 - 4x + 5 = 3(x^2 \_\_\_\_) + 5$	$x^2 - \frac{4}{3}x$
76	$= 3(x^2 - \frac{4}{3}x + \frac{4}{9}) - \_\_\_\_ + 5$	$-\frac{4}{3}$
	$= 3(x - \frac{2}{3})^2 + \frac{11}{3}$	

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17-4. Quadratic Equations

An equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$, is called a quadratic equation.

	Solve the following quadratic equations.	
1	$2x^2 - 6 = 0$ _____	$(\sqrt{3}, -\sqrt{3})$
2	$y^2 - 4y - 7 = 0$ _____ [Hint: Complete the square.]	$(2+\sqrt{11}, 2-\sqrt{11})$
3	$a^2 - 6a + 3 = 0$ _____	$(3+\sqrt{6}, 3-\sqrt{6})$
4	$w^2 = 9 - 2w$ _____ [Hint: Rewrite in the form $ax^2 + bx + c = 0$.]	$(-1+\sqrt{10}, -1-\sqrt{10})$
5	$x^2 - 12x + 40 = 0$ _____	ϕ [An equivalent equation is: $(x-6)^2+4 = 0$.]
6	$x^2 + 5x - 14 = 0$	$(-7, 2)$

Notice that the equations in Items 2, 3, 4, 5 involve polynomials which cannot be factored over the integers. In these equations completing the square is indicated. In Item 6, you might have used completing the square, or you might have factored the polynomial by inspection.

	Consider the quadratic equation $$2x^2 + 3x - 1 = 0.$$	
7	The polynomial $2x^2 + 3x - 1$ _____ (can, cannot) be factored over the integers.	cannot
	However, we can solve this equation by completing the square. In order to do so, we begin by recognizing that	
8	$2x^2 + 3x - 1 = 0$ $x^2 + \_\_\_ - \frac{1}{2} = 0$	$x^2 + \frac{3}{2}x - \frac{1}{2} = 0$
	are equivalent equations.	
9	(We obtain the second equation by multiplying both sides of the first by _____.)	$\frac{1}{2}$

	We have: $2x^2 + 3x - 1 = 0$	
10	$x^2 + \frac{3}{2}x - \frac{1}{2} = \underline{\qquad}$	0
11	$x^2 + \frac{3}{2}x + \underline{\qquad - \qquad} - \frac{1}{2} = 0$	$+\frac{9}{16} - \frac{9}{16}$
12	$\underline{(x + \qquad)^2 - \qquad} = 0$	$(x+\frac{3}{4})^2 - \frac{17}{16}$
13	$(x + \frac{3}{4})^2 - (\frac{\sqrt{17}}{\Box})^2 = 0$	$(\frac{\sqrt{17}}{4})^2$
14	$(x + \frac{3}{4} + \frac{\sqrt{17}}{4})\underline{(x \qquad\qquad)} = 0$	$(x + \frac{3}{4} - \frac{\sqrt{17}}{4})$
	From this chain of equivalent equations we can conclude: the truth set of $2x^2 + 3x - 1 = 0$ is	
15	$\{\frac{-3 - \sqrt{17}}{4}, \underline{\qquad\qquad}\}$	$\frac{-3 + \sqrt{17}}{4}$

If you had trouble with Items 10 to 15 do Items 16 to 24. If not, go to Item 25.

	Suppose we wish to complete the following: $x^2 + \frac{3}{2}x + \qquad = (\qquad)^2.$	
16	We must add to the left side the square of ______ the coefficient of x.	$\frac{1}{2}$
17	$\frac{1}{2}(\frac{3}{2}) = \underline{\qquad}$, and $(\frac{3}{4})^2 = \underline{\qquad}$.	$\frac{3}{4}$, $\frac{9}{16}$
18	$x^2 + \frac{3}{2}x + \frac{9}{16} = \underline{(\qquad)}^2$	$(x + \frac{3}{4})^2$
	You should verify for yourself that $(x + \frac{3}{4})^2 = x^2 + \frac{3}{2}x + \frac{9}{16}$.	
	We had $x^2 + \frac{3}{2}x - \frac{1}{2} = 0$ (Item 8).	
	Using the method of completing the square, we write:	
19	$(x^2 + \frac{3}{2}x + \underline{\qquad}) - \frac{9}{16} - \frac{1}{2} = 0.$	$\frac{9}{16}$
20	Since we added $\frac{9}{16}$, we also ______ $\frac{9}{16}$.	subtracted

To complete Item 12, you had to notice that

21 $-\frac{9}{16} - \frac{1}{2} =$ _____. $-\frac{17}{16}$

To complete Item 13, you had to notice that

22 $\sqrt{\frac{17}{16}} = \frac{\sqrt{17}}{\sqrt{16}} = \frac{\sqrt{17}}{\square}$. $\frac{\sqrt{17}}{4}$

Item 13, when completed, reads: $(x + \frac{3}{4})^2 - (\frac{\sqrt{17}}{4})^2 = 0$.

23 This is the ______ of two squares. difference

Hence it can be factored. One factor is the sum of

24 $x + \frac{3}{4}$ and $\frac{\sqrt{17}}{4}$. The other factor is the ______ of difference

$x + \frac{3}{4}$ and $\frac{\sqrt{17}}{4}$.

Now go through Items 10 to 15 again. Be sure you understand each step.

Consider the quadratic equation $2x^2 + 4x - 3 = 0$.

We apply the method of the preceding example.

Again the first step is to multiply both sides of the

25 equation by ______. We obtain $\frac{1}{2}$

26 ______________ $x^2 + 2x - \frac{3}{2} = 0$

Try to complete the solution for yourself. The truth

27 set is ______. $\{-1-\frac{\sqrt{10}}{2}, -1+\frac{\sqrt{10}}{2}\}$ or $\{\frac{-2-\sqrt{10}}{2}, \frac{-2+\sqrt{10}}{2}\}$

If you had trouble, do Items 28 to 37. If not, go to Item 38.

We want to solve $2x^2 + 4x - 3 = 0$.

$$2x^2 + 4x - 3 = 0$$

$$x^2 + 2x - \frac{3}{2} = 0$$

28 $(x^2 + 2x + \_\_\_) - \_\_\_ - \frac{3}{2} = 0$ 1, 1

29 $(x + \_\_\_)^2 - \_\_\_ = 0$ $(x + 1)^2 - \frac{5}{2}$

	Notice that $\sqrt{\frac{5}{2}} = \frac{\sqrt{5}}{\sqrt{2}}$	
	$= \frac{\sqrt{5}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$	
30	$= \frac{\square}{2}$.	$\frac{\sqrt{10}}{2}$
	Thus $(x + 1)^2 - \frac{5}{2} = 0$	
31	may be written $(x + 1)^2 - (\quad)^2 = 0$.	$(\frac{\sqrt{10}}{2})^2$
32	We then have: $(x + 1 + \frac{\sqrt{10}}{2})(x \underline{\qquad\qquad}) = 0$.	$(x + 1 - \frac{\sqrt{10}}{2})$
	This last sentence is equivalent to	
33	$x + 1 + \frac{\sqrt{10}}{2} = 0$ ______ (and, or) $x + 1 - \frac{\sqrt{10}}{2} = 0$	or
	which is equivalent to	
34	$x = -1 - \frac{\sqrt{10}}{2}$ or $x =$ ______.	$-1 + \frac{\sqrt{10}}{2}$
35	$-1 - \frac{\sqrt{10}}{2} = \frac{\square}{2}$	$\frac{-2 - \sqrt{10}}{2}$
36	Likewise, $-1 + \frac{\sqrt{10}}{2} =$ ______	$\frac{-2 + \sqrt{10}}{2}$
37	The truth set of $2x^2 + 4x - 3 = 0$ is ______.	$\{\frac{-2-\sqrt{10}}{2}, \frac{-2+\sqrt{10}}{2}\}$

	Solve:	
38	$3x^2 + 6x - 1 = 0$ $\{\quad,\quad\}$	$\{\frac{-3-2\sqrt{3}}{3}, \frac{-3+2\sqrt{3}}{3}\}$
39	$x^2 - 6x + 10 = 0$ ______	$\emptyset$ [An equivalent equation is: $(x-3)^2+1 = 0$]

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In this section we have solved several quadratic equations, including:

Equation	Truth set	
$2x^2 - 6 = 0$	$\{\sqrt{3}, -\sqrt{3}\}$	(Item 1)
$x^2 - 12x + 40 = 0$	$\emptyset$	(Item 5)
$x^2 + 5x - 14 = 0$	$\{-7, 2\}$	(Item 6)
$2x^2 + 3x - 1 = 0$	$\{\frac{-3 - \sqrt{17}}{4}, \frac{-3 + \sqrt{17}}{4}\}$	(Item 15)
$3x^2 + 6x - 1 = 0$	$\{\frac{-3 - 2\sqrt{3}}{3}, \frac{-3 + 2\sqrt{3}}{3}\}$	(Item 38)
$x^2 - 6x + 10 = 0$	$\emptyset$	(Item 39)

Each of these equations has the general form

$$ax^2 + bx + c = 0.$$

For example, we recognize that $2x^2 + 3x - 1 = 0$

40 has this form, and that $a =$ ____, $b =$ ____, $c =$ ____. — 2, 3, -1

Notice that in each of the equations the coefficent of x^2 is different from 0.

We have called an equation of the form $ax^2 + bx + c = 0$,

41 where $a \neq 0$, a ______ equation. — quadratic

Suppose someone asks you to solve an equation of the form $ax^2 + bx + c = 0$. You _may_ find that you can factor the polynomial $ax^2 + bx + c$ by inspection. On the other hand, this may not be possible. However, you can _always_ apply the method of completing the square.

Let us apply the method of completing the square to solving $ax^2 + bx + c = 0$.

(Remember, a, b, c are real numbers and $a \neq 0$.)

We may write the chain of equivalent equations:

$$ax^2 + bx + c = 0$$

42 $$x^2 + \frac{b}{a}x + \_\_\_\_ = 0$$ — $\frac{c}{a}$

43 $$x^2 + \frac{b}{a}x + \underline{(\quad)^2 - (\quad)^2} + \frac{c}{a} = 0$$ — $(\frac{b}{2a})^2 - (\frac{b}{2a})^2$

44 $\left(x + \underline{\quad\quad}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$ — $\left(x + \frac{b}{2a}\right)^2$

45 Note that we have used the fact that $\left(\frac{b}{2a}\right)^2 = \underline{\quad\quad}$. — $\frac{b^2}{4a^2}$

(If you had trouble with any of these items look back at Items 8 to 24, where the steps for solving $2x^2 + 3x - 1 = 0$ are shown in detail.)

In Item 44, we obtained the equation

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0.$$

We know that the difference of two squares can be factored. Thus we would like to write

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}$$

as the difference of two squares.

In order to do this, we first write

$$-\frac{b^2}{4a^2} + \frac{c}{a}$$

as a single fraction.

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$$-\frac{b^2}{4a^2} + \frac{c}{a} = -\frac{b^2}{4a^2} + \frac{\square}{\underline{4a^2}}$$

$$= \frac{-b^2 + 4ac}{4a^2}$$

$$= -\frac{b^2 - 4ac}{4a^2}$$

— $\frac{4ac}{4a^2}$

Thus we can rewrite the equation in Item 44:

47 $$\left(x + \frac{b}{2a}\right)^2 - \frac{\square}{\underline{4a^2}} = 0.$$ — $\frac{b^2 - 4ac}{4a^2}$

We can factor

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}$$

as the difference of two squares if and only if

$$\frac{b^2 - 4ac}{4a^2}$$

is the square of a real number. We recall that every non-negative real number is the square of a real number. We recall also that no negative real number is the square of a real number. Thus whether or not we can factor

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}$$

depends on whether or not $\frac{b^2 - 4ac}{4a^2}$ is non-negative.

$4a^2 > 0$ for all real values of a except 0. (If $a = 0$, the original equation is not a quadratic equation.) Thus we see: $\frac{b^2 - 4ac}{4a^2}$ is non-negative if and only if $b^2 - 4ac \geq 0$.

If $b^2 - 4ac \geq 0$, then so is $\frac{b^2 - 4ac}{4a^2}$.

In this case, $\frac{b^2 - 4ac}{4a^2}$ is the square of a non-negative real number, and

48 $$\sqrt{\frac{b^2 - 4ac}{4a^2}} = \underline{\sqrt{\frac{b^2 - 4ac}{(\quad)^2}}}$$ — $\sqrt{\frac{b^2 - 4ac}{(2a)^2}}$

Thus if $b^2 - 4ac \geq 0$, we may write

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} = 0$$

49 as $$\left(x + \frac{b}{2a}\right)^2 - \left(\underline{\frac{\square}{2a}}\right)^2 = 0$$ — $\left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2$

We see that we have the difference of two squares. Hence we have:

50 $$\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \underline{\qquad}\right) = 0$$ — $\frac{\sqrt{b^2 - 4ac}}{2a}$

An equivalent sentence is:

$$x + \frac{b + \sqrt{b^2 - 4ac}}{2a} = 0 \quad \text{or} \quad x + \frac{b - \sqrt{b^2 - 4ac}}{2a} = 0$$

The truth set of this sentence is:

51 $\{\frac{- \sqrt{b^2 - 4ac}}{2a}$, ______ $\}$ — $\frac{-b+\sqrt{b^2-4ac}}{2a}$

We have seen that

$$ax^2 + bx + c = 0$$

$$\text{and} \quad (x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a^2} = 0$$

are equivalent equations. If $b^2 - 4ac \geq 0$ they both have the truth set $\{\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}\}$.

What is the situation if $b^2 - 4ac < 0$?

$a \neq 0$, and hence $4a^2 > 0$.

In this case, $\frac{b^2 - 4ac}{4a^2}$ is negative, and hence

52 $-\frac{b^2 - 4ac}{4a^2}$ is ______. — positive

Notice that for every real value of x,

$$(x + \frac{b}{2a})^2 \geq 0$$

is a true sentence.

Hence for every real value of x the sum of $(x + \frac{b}{2a})^2$

53 and a positive number is ______ (positive, negative). — positive

We may conclude: If $b^2 - 4ac < 0$, then there is <u>no</u> real value of x for which

$$(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a^2} = 0$$

is a true sentence.

When $b^2 - 4ac < 0$, the truth set of $(x + \frac{b}{2a}) - \frac{b^2 - 4ac}{4a^2} = 0$, and hence of the equivalent sentence $ax^2 + bx + c = 0$,

54 is ______. — $\emptyset$

Although the work in Items 42 to 55 may have seemed difficult to you, you should recognize easily that it is simply a generalization of our method of solving quadratics.

We may summarize:

55 1) An equation of the form $ax^2 + bx + c = 0$, when $a \neq 0$, is a ______ equation. | quadratic

2) By completing the square, we can find: If $b^2 - 4ac \geq 0$, then the solutions of the equation are:

$$\frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

56 If $b^2 - 4ac < 0$, then the solution set of the equation is ______. | $\emptyset$

In 2), we have expressions for the solutions of the quadratic equation $ax^2 + bx + c = 0$ in terms of the coefficients, in the case where $b^2 - 4ac \geq 0$. These expressions for the solutions are often referred to as the <u>quadratic formula</u>.

In order to show an application of the quadratic formula we will use it to solve the equation $2x^2 + 3x - 1 = 0$. (This equation was solved by another method in Items 7 to 15.)

The equation $2x^2 + 3x - 1 = 0$ is of the form

57 $ax^2 + bx + c = 0$, where $a =$ ____, $b =$ ____, $c =$ ____. | 2, 3, -1

58 $b^2 - 4ac = (\quad)^2 - 4(\quad)(\quad) = 17$ | $(3)^2 - 4(2)(-1)$

59 Since $b^2 - 4ac >$ ______, | 0

the equation has solutions of the form:

$$\frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

The solutions are

60 $$\frac{-(\quad) - \sqrt{17}}{2(2)} \quad \text{and} \quad \frac{-(\quad) + \sqrt{17}}{2(\quad)}$$ | $\frac{-(3)-\sqrt{17}}{2(2)}, \frac{-(3)+\sqrt{17}}{2(2)}$

That is, they are: $\frac{-3 - \sqrt{17}}{4}$ and $\frac{-3 + \sqrt{17}}{4}$

You should verify that this is the solution we found in Item 15.

Consider the quadratic equation $x^2 - 6x + 10 = 0$.

It is of the form $ax^2 + bx + c$, with

61 $a =$ ____, $b =$ ____, $c =$ ____. 1, -6, 10

62 For this equation, $b^2 - 4ac = (\quad)^2 - 4(\quad)(\quad)$ $(-6)^2 - 4(1)(10)$

63 $= \_\_\_\_$ -4

We may conclude that the truth set of this equation

64 is ______, since $b^2 - 4ac < 0$. $\emptyset$

Note that this is the result obtained in Item 40.

If you wish, you may practice applying the quadratic formula by using it to solve the equations in Items 5, 6, 38, 39.

Let us conclude this section with some problems which lead to quadratic equations.

The square of a number is 7 greater than 6 times the number. What is the number?

Let n represent the number.

An open sentence is:

65 $$n^2 = \_\_\_\_\_\_\_\_.$$ $n^2 = 7 + 6n$

We see that

$$n^2 = 7 + 6n$$

66 $$n^2 - 6n - 7 = \_\_\_\_.$$ 0

$n^2 - 6n - 7 = 0$ is a quadratic equation. In order to solve it, you could try to factor the polynomial $n^2 - 6n - 7$ by inspection.

67 $$n^2 - 6n - 7 = (\quad)(\quad).$$ $(n - 7)(n + 1)$

68 Hence the truth set of $n^2 - 6n - 7 = 0$ is ______. $\{7, -1\}$

Recall that as a final step you should check in the original problem.

69 Do both the numbers 7, -1 fit the original problem? ______ (yes,no) — yes

70 Note that this problem has ______ (how many) solutions. — two

In this example, we were able to solve the quadratic equation $n^2 - 6n - 7$ by factoring $n^2 - 6n - 7$ by inspection. You might have preferred to use the method of completing the square in solving this equation. If you had wished to familiarize yourself with the quadratic formula you would have chosen to solve the quadratic equation by this method.

The length of a rectangle is 5 inches more than its width. Its area is 84 square inches. Find the width.

Solve this problem. Then use your work to complete the items below.

71 If w represents the width in inches of the rectangle, an appropriate open sentence is: ______ = 84. — $w(w + 5) = 84$

Since w represents the width in inches of a rectangle, we should consider as domain of this open sentence the set of <u>positive</u> real numbers.

72 An equivalent open sentence is ______ = 0. — $w^2 + 5w - 84 = 0$

73 The truth set of the equation $w^2 + 5w - 84 = 0$ is ______. — $\{7, -12\}$

74 Only one of the numbers 7, -12 is positive. Hence only one value for the width--namely, ______ inches--satisfies the conditions of the problem. — 7

17-5. Summary and Review

In this chapter we have been particularly concerned with quadratic polynomials--that is, polynomials of the form $ax^2 + bx + c$, where $a \neq 0$. If a, b, c are integers, then $ax^2 + bx + c$ is a polynomial over the integers, and it is often useful to ask whether the polynomial can be factored over the integers.

We recall that a first step in factoring a polynomial is to see whether each term has a common factor. If so, we can apply the distributive property directly.

In order to factor $ax^2 + bx + c$ over the integers by inspection, we look for integers r, s, m, n such that

$$(rx + m)(sx + n) = ax^2 + bx + c.$$

If $a = 1$, we need only find integers m and n such that

$$(x + m)(x + n) = x^2 + bx + c.$$

We recall that the method of completing the square may also be used in finding a factorization over the integers of a polynomial of the form $x^2 + bx + c$.

We have observed that in factoring a quadratic polynomial over the real numbers the method of completing the square also applies.

We have seen that every quadratic polynomial $ax^2 + bx + c$ can be written in the form

$$a(x - h)^2 + k.$$

We have also seen that we can find the solution set for every quadratic equation

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

by completing the square. (We may find that the solution set is the null set.) If we apply the method of completing the square to the equation $ax^2 + bx + c = 0$, we can obtain a general formula, the quadratic formula, which gives complete information about the solution set of the quadratic equation in terms of a, b, c.

Use the distributive property to factor (if possible) each of the following polynomials over the integers.

1	$15a^2 - 30b = 15(\_\_\_\_\_)$	$15(a^2 - 2b)$
2	$a^3 - 2a^2 + 3a = a(\_\_\_\_\_)$	$a(a^2 - 2a + 3)$
3	$(6r^2s)x - (6r^2s)y = \_\_\_\_\_$	$6r^2s(x - y)$

4 | $6p - 12q + 30 =$ ______ | $6(p - 2q + 5)$

5 | $6p - 12q + 31 =$ ______ | cannot be factored

6 Which of the following is not a correct factorization?

[A] $9y^2 - 4 = (3y + 2)(3y - 2)$

[B] $3x^3y^2 - 2x^2y^3 = (3x - 2y)x^2y^2$

[C] $a(c + d) - b(c + d) = (a - b)(c + d)$

[D] $x^2 - 15x + 25 = (x - 5)^2$

Since $(x - 5)^2 = (x - 5)(x - 5)$

$= x^2 - 10x + 25,$

[D] is the proper choice.

Write the result of performing the multiplications. Answers are on page xx.

7. $(x + 3)^2 =$ ______

8. $(\cdot - 2)^2 =$ ______

9. $(x + \sqrt{2})^2 =$ ______

10. $(a + b)^2 =$ ______

11. $(x - y)^2 =$ ______

12. $\big((x - 1) + a\big)\big((x - 1) - a\big) =$ ______

13. $(100 + 1)^2 =$ ______

14. $(2x - 3y)^2 =$ ______

15. $(3a + 4b)^2 =$ ______

Factor each of the following over the integers if possible. Answers are on page xx.

16. $x^2 - 4$

17. $x^2 + 4$

18. $4t^2 + 12t + 9$

19. $4z^2 - 20z + 25$

20. $9s^2 + 65t + 4t^2$

21. $2a^2 - 20ab + 5b^2$

22. $rs + 25t + 3rt$

23. $x^2 - 4b - 4x + bx$

24. $9ma + 8b + 6ab + 12m$

25. $2a^3 - 20a^2b + 50ab^2$

26. $z^4 + 16z^2 + 64$

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Factor over the real numbers. Answers are on page xx.

27. $n^2 - 10n + 24$

28. $z^2 - 2z + 18$

29. $-x^2 + 7x - 12$

30. $-x^2 - 4x + 12$

31. $-x^2 + x + 12$

32. $a^2 - 16a + 64$

33. $a^2 + 8a + 64$

34. $a^2 - 20a + 64$

35. $a^2 - 16a - 64$

36. $d^2 - 2$

37. $h^2 - 169$

38. $10a + 39 + a^2$

39. $5a^3 - 15a^2 + 30a$

40. $7x^2 - 63$

Factor over the real numbers.

41	$6x^2 - 144x - 150 =$ ______	$6(x - 25)(x + 1)$
42	$6x^2 + 60x + 150 =$ ______	$6(x + 5)^2$
43	$6x^2 + 25x + 150 =$ ______	not factorable
44	$6x^2 - 87x + 150 =$ ______	$3(x - 2)(2x - 25)$
45	$6x^2 + 63x - 150 =$ ______	$3(x - 2)(2x + 25)$

* Sometimes we find that our knowledge of prime factorizations of the integers helps us factor a polynomial by inspection over the integers. The following polynomials can be factored over the integers. Find the factorization.

*46	$6x^2 - 61x + 150 =$ ______	$(x - 6)(6x - 25)$
*47	$6x^2 + 65x + 150 =$ ______	$(3x + 10)(2x + 15)$
*48	$6x^2 - 11x - 150 =$ ______	$(x - 6)(6x + 25)$
*49	$6x^2 + 7x - 24 =$ ______	$(2x - 3)(3x + 8)$

Write in the form $a(x - h)^2 + k$:

50	$x^2 + 2x - 2 = (\quad)^2 -$ ______	$(x + 1)^2 - 3$
51	$x^2 + 8x + 3 =$ ______	$(x + 4)^2 - 13$
52	$2x^2 + 8x - 5 =$ ______	$2(x + 2)^2 - 13$

Find the truth set of:

		Answer
53	$a^2 - 5a + 6 = 0$ ______	$\{2, 3\}$
54	$x^2 + 6x = 0$ ______	$\{0, -6\}$
55	$x^2 = 2x + 1$ ______ [Hint: Complete the square.]	$\{1 - \sqrt{2}, 1 + \sqrt{2}\}$
56	$6x^2 + 6x - 72 = 0$ ______	$\{-4, 3\}$
57	$x^2 + 6 = 7x$ ______ [Hint: Rewrite in form $ax^2 + bx + c = 0$.]	$\{1, 6\}$
58	$(x - 2)(x + 1) = 4$ ______ [Be cautious!]	$\{-2, 3\}$
59	$x^2 + 2x + 5 = 0$ ______	$\emptyset$
60	$3x^2 + 6x - 1 = 0$ ______	$\{\frac{-3-2\sqrt{3}}{3}, \frac{-3+2\sqrt{3}}{3}\}$

In each of the following you are to translate the given situation into an open sentence, find the truth set, and answer the questions asked. Remember, you should check by determining that the solution obtained satisfies all the conditions of the original problem. Answers are shown on page xxi.

61. The square of a number is 9 less than 10 times the number. What is the number?

62. A rectangular bin is 2 feet deep and the perimeter of its base is 24 feet. If the volume of the bin is 70 cubic feet, what are the length and the width of the bin?

63. Two plywood panels, each of which cost 30 cents per square foot, were found to have the same area, although one of them was a square and the other a rectangle 6 inches longer than the square but only 3 inches wide. What were the dimensions of the two panels?

64. If the length of a rectangle is 7 inches longer than its width and if its diagonal is 13 inches, how wide is the rectangle?

65. The altitude of a triangle is 3 inches shorter than its base, and its area is 14 square inches; how long is the base of the triangle?

66. If the perimeter of a rectangle is 28 feet long and its area is 24 square feet, how long is the rectangle?

67. Starting from the same point Rosemary walked north at a certain constant rate, while Lorraine walked west at a constant rate which was 1 m.p.h. greater than that of Rosemary. If they were 5 miles apart at the end of 1 hour, what was the walking rate of each?

68. The sum of two numbers is 15 and the sum of their squares is 137; find the numbers.

69. One number is 8 less than another, and their product is 84. Find the numbers.

70. The product of two consecutive odd numbers is 15 more than 4 times the smaller number. What are the numbers?

71. The sum of 14 times a number and the square of the number is 11. Find the number.

72. Find the truth set. (Answer is on page xxv.)

$x^2 + 2\sqrt{3}x - 10 = 0$ [Hint: Complete the square.]

73. Prove: The square of an odd integer is odd.

*74. Prove: If m is an odd integer, $m^2 - 1$ is a multiple of 8.

The completed proofs are on page xxv.

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DIVIDING POLYNOMIALS; RATIONAL EXPRESSIONS

18-1. Division of Polynomials

In Chapter 16 we observed that every polynomial could be written in common polynomial form. You have had a great deal of practice in writing products of polynomials in common polynomial form.

As you already know, division and multiplication are closely related. It is natural to ask what we can say about dividing polynomials.

1	We can easily verify that $x^2 - 6x + 8 =$ ( )( ).	$(x - 2)(x - 4)$
	The fact that $(x - 2)(x - 4) = x^2 - 6x + 8$ can be verified by using the distributive property and other properties that are true for all real values of x.	
2	Thus, in Item 1 you completed a sentence that is true for all ______ values of x.	all real values

We know that if n, d, q are real numbers and $d \neq 0$, then $n \div d = q$ means $n = qd$. This leads us to compare the statements

$$x^2 - 6x + 8 = (x - 2)(x - 4)$$
$$(x^2 - 6x + 8) \div (x - 4) = x - 2 .$$

	If x has the value of 7, then	
3	$x^2 - 6x + 8$ has the value ______,	15
4	$x - 4$ has the value ______,	3
5	$x - 2$ has the value ______.	5
6	$15 = 5 \cdot 3$, and $15 \div 3 =$ ______.	5
	If x is 0, then	
7	$x^2 - 6x + 8$ is ______,	8
8	$x - 4$ is ______,	-4
9	$x - 2$ is ______.	-2

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10	$(-2)(-4) =$ ______, and also $8 \div (-4)$ [illegible] ______.
	We have <u>verified</u> that the statement
	$(x^2 - 6x + 8) \div (x - 4)$ $x -$ [illegible]
11	is true if x is 7 and if x is ______.

12	Is the statement
	$(x^2 - 6x + 8) \div (x - 4)$ [illegible]
	true for <u>all</u> real values of x ?
	[A] yes [B] [illegible]

Did you remember that division by 0 is meaningless? [illegible] **replace x by 4 in the statement,** the divisor is [illegible]. [illegible] **the statement is** <u>not</u> true if x is 4. You [illegible]

You should now understand that

$$(x^2 - 6x + 8) \div (x - 4) \; [illegible]$$

is true for all real values of x <u>except</u> 4. [illegible]

$$x^2 - 6x + 8 \; [illegible] \; (x - 4)(x - [illegible]$$

simply states that

$$0 = 0 \cdot [illegible]$$

We cannot write a corresponding statement [illegible].

13	$x^2 - 9 =$ <u>(______)(______)</u>
14	Hence, for all real values of x except ______,
15	$(x^2 - 9) \div (x - 3) =$ ______.
16	Likewise, for all real values of x except ______
17	$(x^2 - 9) \div (x + 3) =$ ______.

We have found:

$$(x^2 - 6x + 8) \div (x - 4) \quad x - [illegible]$$

$$(x^2 - 9) \div (x + 3) \quad x - [illegible]$$

In each of these statements we have written the quotient of two polynomials as a polynomial. Pause a moment to consider the question: If we are given any two polynomials, can their quotient be expressed as a polynomial?

The answer, as you should have decided, is "No".

Is there a polynomial Q such that the product of this
18 polynomial and x is $x + 1$? ______ no

There is no polynomial Q such that

$$(x + 1) \div (x = Q)$$

for all values of x except 0.

19 Notice that $x(2x + 5) =$ ___ + ___ $2x^2 + 5x$

20 $x(x - 1) =$ ______ $x^2 - x$

21 $x(x^2 + x - 4) =$ ______ $x^3 + x^2 - 4x$

If Q is any polynomial, the product of x and Q is
22 a polynomial in which each term has the factor ______. x

Since x is not a factor of 1, we see that $x + 1$ is not the product of a polynomial and x.

23 $(x^2 - 7x + 6) \div (x - 1) =$ ______ if $x \neq 1$. $x - 6$

[Hint: Factor $x^2 - 7x + 6$]

24 $(x^2 + 5x - 6) \div (x - 1) =$ ______ if $x \neq 1$. $x + 6$

25 $(x^2 - 1) \div (x - 1) =$ ______ if $x \neq$. $x + 1$

Consider $(x^2 + 1) \div (x - 1)$.

26 $x - 1$ ______ (is, is not) a factor of $x^2 + 1$. is not

We cannot find a polynomial which is equal to $(x^2 + 1) \div (x - 1)$ for all values of x except 1.

Again we are reminded of familiar facts about integers.

27 We ______ (can, cannot) find an integer q such that $29 = 6q$. cannot

Consequently, we cannot write $29 \div 6$ as an integer.

28 We often write $29 \div 6$ as $\frac{29}{\square}$. — $\frac{29}{6}$

In the fraction $\frac{29}{6}$,

29 29 is the dividend, and 6 the ______. — divisor

As you know, $\frac{29}{6}$ can be written as the mixed number

30 $4\frac{5}{6}$, which means $4 +$ ______. — $4 + \frac{5}{6}$

---

31 In $\frac{171}{23}$, 171 is the ______ (dividend, divisor) and 23 is — dividend

32 the ______ (dividend, divisor). — divisor

Similarly, in the indicated quotient, $\frac{2x^2 - 6x + 9}{x - 3}$,

33 the dividend is ______, — $2x^2 - 6x + 9$

34 and $x - 3$ is the ______. — divisor

If you were given $\frac{171}{23}$, you might write it as the

35 mixed number $7\frac{\square}{23}$. — $7\frac{10}{23}$

In this course we have had little occasion to use mixed numbers. For most purposes $\frac{171}{23}$ is easier to work with than $7\frac{10}{23}$. However, at times, the form $7\frac{10}{23}$ is useful.

$$\frac{171}{23} = 7 + \frac{10}{23}$$

36 $$171 = 7 \times \square + 10$$ — $7 \times 23 + 10$

In Item 36 we have written 171 as the sum of an integer (7), times the divisor (23), and an integer (10) which is less than the divisor.

We sometimes say: When 171 is divided by 23, the

37 quotient is 7 and the remainder is ______. — 10

In general, if we are asked to divide a positive integer n by a positive integer d, we find non-negative integers q and r such that

$$n = qd + r,$$

where the remainder, r, is less than the divisor, d.

Consider $29 \div 6$.

38 $[illegible] = \underline{\square \times [illegible] + \square}$ — $4 \times 6 + [illegible]$

$[illegible] = qd + [illegible]$

39 Here $[illegible]$, $[illegible]$, $q =$ ______, — 4

40 [illegible] ______. [illegible]
the [illegible], [illegible].

Consider $\frac{2953}{13}$.

41 We may write: $2953 = \underline{\square \times 13 + \square}$ — $227 \times 13 + 2$

This is of the form:

$$n = qd + r,$$

42 where q is ______ and r is ______. — 227, 2

43 Notice that the remainder, ______, is less than the divisor, 13. — 2

An alternative statement for

$$n = qd + r$$

is

44 $$\frac{n}{d} = \underline{\square + \frac{\square}{d}}.$$ — $q + \frac{r}{d}$

Since $2953 = 227 \times 13 + 2$, we see that

45 $$\frac{2953}{13} = \underline{227 + \frac{\square}{13}}.$$ — $227 + \frac{2}{13}$

Considering now $105 \div 7$, we see that

46 $105 = \underline{\square \times 7 + 0}$ — $13 \times 7 + 0$

47 Here the remainder is ______. — 0

48 0 is a non-negative integer less than the divisor, ___. — 7

$\frac{8}{17}$ is an indicated quotient.

49 $8 = \underline{\square \times 17 + \square}$ — $0 \times 17 + 8$

50 Here q is ______, and r is ______. — , 8

The remainder, 8, is less than the divisor, 17.

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We now turn to a simple case of division of polynomials.

Consider $\frac{x^2 - 5x + 1}{x}$ $(x \neq 0)$.

51 Obviously, $x^2 - 5x + 1 = \underline{(x - 5)x + \quad\quad}$. — $(x - 5)x + 1$

Let us use

N to represent $x^2 - 5x + 1$,

D to represent x.

Then we observe that

$$x^2 - 5x + 1 = (x - 5)x + 1$$

has the form

$$N = QD + R$$

52 where Q is $\underline{x \quad\quad}$ — $x - 5$

53 and R is ______. — 1

We could also write

54 $$\frac{\square}{x} = x - 5 + \frac{1}{x}$$ — $x^2 - 5x + 1$

which has the form

$$\frac{N}{D} = Q + \frac{R}{D}.$$

55 In this example, notice that N is of degree ______ — 2

56 and D is of degree ______. — 1

The degree of R is 0, which is <u>less than</u> the degree of D.

Of course, that example was easy. Try

$$\frac{2x^2 - 6x + 9}{x - 3}.$$

57 Surely, $2x^2 - 6x + 9 = \underline{2x(x - 3) + \quad\quad}$. — $2x(x - 3) + 9$

58 Here Q is ______ and — $2x$

59 R is ______. — 9

60 The degree of R is ______. — 0

In alternate form:

61 $$\frac{2x^2 - 6x + 9}{x - 3} = 2x + \frac{\square}{x - 3}.$$ — 9

62 Notice that

$$2x^2 - 6x = \underline{2x(\quad\quad)}$$ 2x(x - 3)

We see that this is of the form

$$N = Q \cdot D + R,$$

63 where R is ______. 0

You should recall that the degree of the polynomial 0 is not defined.

These examples show the similarity between the division of polynomials and the division of integers. In each case we wanted to divide a polynomial N by a polynomial D. We were able to find polynomials Q and R such that

$$N = QD + R,$$

and either R was a polynomial with degree less than that of D or R was 0.

In the next section you will learn a procedure which may be used, if you are given polynomials in one variable N and D, with D not 0, to find Q and R such that

$$N = QD + R,$$

where R is 0 or is of degree less than that of D. This process, you will find, is analagous to "long division" for integers.

The division process requires repeated subtraction. Hence, we will conclude this section by practicing some subtraction of polynomials. You will recall that in Section 16-1 we noted briefly that it is sometimes useful to write subtraction problems in vertical form.

Thus, $(-3x^2 + x - 2) - (2x^2 - 3x + 1)$ can be written:

From $-3x^2 + x - 2$

subtract $\underline{2x^2 - 3x + 1}$

64 $-5x^2 + \square - 3$ (difference) $-5x^2 + 4x - 3$

Again, from $a^3 - 5a^2 + 2a + 1$

subtract $\underline{a^3 + 7a^2 + 9a - 11}$

65 $\square$ (difference) $-12a^2 - 7a + 12$

Here is a slightly harder one:

$$(-5x^4 + 2x^3 - x + 1) - (3x^4 - x^2 + x + 2):$$

	From	$-5x^4 + 2x^3 \qquad - x + 1$
	subtract	$\underline{3x^4 \qquad\qquad - x^2 + x + 2}$
66		$\boxed{}$

$-8x^4+2x^3+x^2-2x-1$

Notice how we placed like powers of x in the same column.

	From	$-3x^4 \qquad - 5x^2 - 7x + 2$
	subtract	$\underline{-3x^4 - 2x^3 - 3x^2 \qquad - 6}$
67		$\boxed{}$

$-2x^3-2x^2-7x+8$

Set up the following in vertical form on your response sheet and perform the indicated operation. Check your work with the work shown on page i.

68. Subtract $3a^2 - 6a + 9$ from $3a^2 + 7a - 11$.

69. From $12x^3 - 11x^2 + 3$ subtract $12x^3 + 6x + 9$.

70. Add $14y^2 + 8y - 16$ and $-12y^2 + 3y$.

71. From $-6x + 8$ subtract $-6x - 1$.

18-2. Division of Polynomials, Concluded

Let N and D be polynomials in one variable. We are interested in finding polynomials Q and R such that

$$N = QD + R$$

and either R is 0 or R has degree less than that of D.

We have already noted that this problem is similar to a familiar one involving integers. For example,

$$2953 = 227 \times 13 + 2$$

If we are given 2953 and 13 we can find the appropriate numbers, 227 and 2, by long division.

Examine carefully the long division process displayed below. Be sure you understand how each step is written.

$$\begin{array}{r l}
13\,\overline{)2953}\,\underline{|227} & \\
\underline{2600} = & 200 \times \underline{13} \\
353 & \\
\underline{260} = & 20 \times \underline{13} \\
93 & \\
\underline{91} = & 7 \times \underline{13} \\
2 & 2
\end{array}$$

Thus: $\quad 2953 = (200 \times 13) + (20 \times 13) + (7 \times 13) + 2$

$\qquad\qquad = (200 + 20 + 7)13 + 2$

$\qquad\qquad = 227 \times 13 + 2$

	The display shows that when we divide 2953 by 13	
1	we obtain the quotient _____ and the	227
2	remainder _____.	2
3	Notice that $227 = \underline{2 \times 10^2 + \square \times 10 + \square}$.	$2\times10^2+2\times10+7$
	In dividing 2953 by 13, what we really do is to	
4	subtract multiples of _____ from 2953.	13
	We first subtract 200×13, or 2600. We then subtract 20×13, or 260, and finally	
5	we subtract $\underline{7 \times}$ _____ or _____.	7×13, 91

Now let us look at a procedure for dividing the polynomial $2x^2 + x - 5$ by the polynomial $x - 3$.

$$
\begin{array}{r|l|l}
x - 3 & 2x^2 + x - 5 & 2x + 7 \\
\hline
& \underline{2x^2 - 6x} & = 2x(x - 3) \\
& 7x - 5 & \\
& \underline{7x - 21} & = 7(x - 3) \\
& 16 & = 16
\end{array}
$$

Thus:

$$
\begin{aligned}
2x^2 + x - 5 &= 2x(x - 3) + 7(x - 3) + 16 \\
&= (2x + 7)(x - 3) + 16
\end{aligned}
$$

$$\frac{2x^2 + x - 5}{x - 3} = 2x + 7 + \frac{16}{x - 3}$$

6 In this example, we are dividing $2x^2 + x - 5$ by ____. **$x - 3$**

This division problem is written

$$\begin{array}{r|l|l} x - 3 & 2x^2 + x - 5 & \_\_\_\_ \\ \hline \end{array}$$

7 Our first step is to think: since $x \cdot$ ______ $= 2x^2$, we should write **$2x$**

8 $$\begin{array}{r|l|l} x - 3 & 2x^2 + x - 5 & \square \\ \hline \end{array}$$ **$2x$**

Now multiply $x - 3$ by $2x$; and write

$$\begin{array}{r|l|l} x - 3 & 2x^2 + x - 5 & 2x \\ \hline & \underline{2x^2 - \square} & \end{array}$$

9 **$2x^2 - 6x$**

Now we subtract, and write

$$\begin{array}{r|l|l} x - 3 & 2x^2 + x - 5 & 2x \\ \hline \text{(subtract)} & \underline{2x^2 - 6x} & \\ & \underline{\square - \square} & \end{array}$$

10 **$7x - 5$**

We now repeat the steps of multiplying $x - 3$ by a

11 suitable expression, and then ______ing. **subtracting**

Since the result of the last subtraction was $7x - 5$, we think:

Multiply the divisor $x - 3$ by 7 and write

12 $7x - 21$ since $7(\_\_\_\_) = 7x - 21$. — $7(x - 3)$

13 Finally, we obtain 16 when we subtract ______ — $7x - 21$

14 from ______. — $7x - 5$

When we have finished, our work looks like this:

$$\begin{array}{r|l|l} x - 3 & 2x^2 + x - 5 & 2x + 7 \\ & \underline{2x^2 - 6x} & \\ & \quad\quad 7x - 5 & \\ & \quad\quad \underline{7x - 21} & \\ & \quad\quad\quad\quad 16 & \end{array}$$

15 The quotient Q is seen to be ______. — $2x + 7$

16 The remainder R is ______. — 16

17 And $\dfrac{2x^2 + x - 5}{x - 3} =$ ______. — $2x + 7 + \dfrac{16}{x - 3}$

Here is a brief summary of what we have done. To divide $2x^2 + x - 5$ by $x - 3$ we:

A. Subtract a multiple of $x - 3$ to eliminate the x^2 term.

B. Subtract another multiple of $x - 3$ to eliminate the x term.

The result of this last subtraction, 16, is of degree 0, which is lower than the degree of the divisor. Thus we have finished the division.

On the response sheet try dividing $x^3 + 3x^2 - 38x - 10$ by $x - 5$. Your first step looks like this:

18 $$\begin{array}{r|l|l} x - 5 & x^3 + 3x^2 - 38x - 10 & \square \end{array}$$ — x^2

We select x^2 since when we subtract this multiple of $x - 5$, we eliminate the term x^3.

Complete the problem on your response sheet and then compare your work with that on page i.

The result of the preceding example shows that $x - 5$ is a factor of $x^3 + 3x^2 - 38x - 10$. This is so because the remainder after division by $x - 5$ is 0.

Let us try another one. On your response sheet divide the polynomial $5x^2 + 3x - 3$ by $x - 2$. Compare your work with that on page i.

19 Your result shows that $\frac{5x^2 + 3x - 3}{x - 2} =$ ______. $5x + 13 + \frac{23}{x - 2}$

We also see that:

20 $5x^2 + 3x - 3 = (\quad)(\quad) +$ ______. $(5x+13)(x-2)+23$

Perform the indicated divisions on scratch paper using the form we have discussed.

21 $\frac{2x^2 - 5x + 3}{x - 2} =$ ______ $2x - 1 + \frac{1}{x - 2}$

22 $\frac{3x^2 + 7x + 5}{x + 3} =$ ______ $3x - 2 + \frac{11}{x + 3}$

23 $\frac{2x^3 + 5x^2 - x + 2}{x + 1} =$ ______ $2x^2+3x-4 + \frac{6}{x+1}$

24. Divide $\frac{2x^3 + 2x^2 + 5}{x - 6}$. (Hint: Write the dividend as $2x^3 + 2x^2 + 0x + 5$.) Check with the work on page i.

Let's try another problem of division.

25 $\frac{x^2 - 2x - 15}{x - 5} =$ ______ $x + 3$

26 The remainder is ______. 0

27 Thus, $x^2 - 2x - 15 = (\quad)(x - 5)$. $(x + 3)$

We could say that $x^2 - 2x - 15$ is a multiple of $x - 5$,

28 or that $x - 5$ is a ______ of $x^2 - 2x - 15$. factor

Notice that we could also have found the quotient $x + 3$ by factoring $x^2 - 2x - 15$.

29 $3x + 1$ ______ (is, is not) a factor of $3x^3 - 2x^2 + 14x + 5$. is

You should not have guessed for the response above. You should have divided to obtain:

30 $\frac{3x^3 - 2x^2 + 14x + 5}{3x + 1} =$ ______. $x^2 - x + 5$

31 $(3x + 1)$ and $(x^2 - x + 5)$ are ______ of $3x^3 - 2x^2 + 14x + 5$. factors

32	Divide $x^3 - 3x^2 + 7x - 1$ by $x - 3$.	$x^2 + 7 + \frac{20}{x-3}$
33	Divide $x^2 - 2x + 15$ by $x - 5$.	$x + 3 + \frac{30}{x-5}$
34	Divide $x^4 - 9x^2 - 1$ by $x + 3$.	$x^3 - 3x^2 + \frac{-1}{x+3}$

Perform the indicated divisions:

35	$\frac{4x^2 - 4x - 15}{2x - 3}$ ______		$2x + 1 + \frac{-12}{2x-3}$
36	$\frac{6x^3 - x^2 - 5x + 4}{3x - 2}$ ______		$2x^2 + x - 1 + \frac{2}{3x-2}$
37	$\frac{x^4 - 1}{x - 1}$ ______	[Hint: Write $x^4 - 1$ as $x^4 + 0x^3 + 0x^2 + 0x - 1$.]	$x^3 + x^2 + x + 1$
38	$\frac{x^4 + 1}{x + 1}$ ______		$x^3 - x^2 + x - 1 + \frac{2}{x+1}$
39	$\frac{x^5 + 1}{x + 1}$ ______		$x^4 - x^3 + x^2 - x + 1$

Perform the indicated divisions:

40	$\frac{2x^2 - 4x + 3}{x - 3}$ ______	$2x + 2 + \frac{9}{x-3}$
41	$\frac{4x^2 - 4x - 13}{2x + 3}$ ______	$2x - 5 + \frac{2}{2x+3}$
42	$\frac{2x^3 - 5x^2 - 8x + 9}{2x + 3}$ ______	$x^2 - 4x + 2 + \frac{3}{2x+3}$
43	$\frac{x^3 + x - 2}{x + 2}$ ______	$x^2 - 2x + 5 + \frac{-12}{x+2}$

* To perform the indicated division $\frac{x^2 + x - 1}{2x + 1}$, we begin by writing

*44 $2x + 1 \overline{\big) x^2 + x - 1}$ $\square$ — $\frac{1}{2}x$

*45 since $(\frac{1}{2}x)(2x) =$ ______. — x^2

Completing the problem we see:

*46 $\frac{x^2 + x - 1}{2x + 1} =$ ______. — $\frac{1}{2}x + \frac{1}{4} - \frac{\frac{5}{4}}{2x + 1}$

Perform the indicated divisions:

*47 $\dfrac{3x^2 + 2x}{2x + 1}$ ______ $\frac{3}{2}x + \frac{1}{4} - \dfrac{\frac{1}{4}}{2x+1}$

*48 $\dfrac{3x^3 + 7x^2 - 5x + 4}{3x - 1}$ ______ $x^2 + \frac{8}{3}x - \frac{7}{9} + \dfrac{\frac{29}{9}}{3x-1}$

Perform the indicated divisions:

*49 $\dfrac{2x^3 - x^2 + 7x - 1}{x^2 - 3}$ ______ $2x - 1 + \dfrac{13x - 4}{x^2 - 3}$

*50 $\dfrac{x^3 - 2x^2 + 7x - 1}{x^2 - 2x - 1}$ ______ $x + \dfrac{8x - 1}{x^2 - 2x - 1}$

Obtain the second factor in each of the following:

51 $9x^6 - 25x^4 + 3x + 5 = (3x + 5)(\_\_\_\_)$ $3x^5 - 5x^4 + 1$

52 $x^9 + 1 = (x^3 + 1)(\_\_\_\_)$ $x^6 - x^3 + 1$

*53 $2x^4 - 5x^2 - x + 1 = (x^2 - x - 1)(\_\_\_\_)$ $2x^2 + 2x - 1$

18-3. Products and Quotients Involving Polynomials

We have already had occasions to observe certain similarities between our conclusions about polynomials and the properties of integers. It is reasonable to expect that our knowledge about factoring polynomials will help us handle fractions involving polynomials, just as our knowledge about factoring integers helps when we work with rational numbers.

We must be sure we understand the meaning of a fraction in which the numerator and denominator are polynomials.

For example, let us consider $\dfrac{3x + 2}{x - 1}$.

1 If x is 3, then $\dfrac{3x + 2}{x - 1}$ is ______. $\frac{11}{2}$

2 If x is $-\frac{5}{2}$, then $\frac{3x + 2}{x - 1}$ is ______. $\frac{11}{7}$

3 Does $\frac{3x + 2}{x - 1}$ name a real number for all real values of x? ______ no

4 No! The domain of x must exclude ______. 1

The domain of the variable must exclude values for which the denominator has the value 0.

For each of the following indicated quotients of polynomials, complete the items to indicate that certain values of the variable are excluded.

5 $\frac{3x - 2}{x}$, $x \neq$ ______ $x \neq 0$

6 $\frac{[illegible]}{5 + x}$, $x \neq$ ______ $x \neq -5$

7 $\frac{4x - 1}{2x + 2}$, ______ $x \neq -1$

8 $\frac{1}{y^2 - 9}$, $y \neq$ ______ and $y \neq$ ______ $y \neq 3$ and $y \neq -3$

9 $\frac{x}{y}$, ______ $y \neq 0$ [$x = 0$ is permitted.]

10 $\frac{1 - x}{x^2 - 2x - 3}$, ______ $x \neq 3$ and $x \neq -1$

Notice in Item 10 that your ability to factor $x^2 - 2x - 3$ was useful.

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11 Which of the following names a real number for all values of the indicated variables?

[A] $\frac{3x^2 - 2x + 1}{x}$

[B] $\frac{x + y}{x - y}$

[C] $\frac{x + y}{x + y}$

[D] $\frac{x + y}{(x + y)^2}$

[E] $\frac{x + 4}{x^2 + 16}$

[A] is not a real number if x is 0. [B] is not a real number if $x = y$. For example, [B] is not a real number if x is 1 and y is 1. Neither [C] nor [D] is a real number if $x = y$. [E] is the correct choice, since $x^2 + 16$ is positive for all values of x, and hence is not 0 for any real value of x.

We have observed (Chapter 13) that in order to write the common name for a rational number expressed as a fraction it is often helpful to factor the numerator and denominator.

For example, to write the simplest name for $\frac{12}{30}$, we can proceed as follows:

12 $\frac{12}{30} = \frac{2^2 \cdot \square}{2 \cdot \square \cdot \square}$ — $\frac{2^2 \cdot 3}{2 \cdot 3 \cdot 5}$

13 $= \frac{2}{5} \cdot \frac{2 \cdot 3}{\square \cdot \square}$ — $\frac{2 \cdot 3}{2 \cdot 3}$

14 $=$ ______ (common name) — $\frac{2}{5}$

(You may not need to write each step.)

We were able to observe, by factoring 12 and 30,

15 that the greatest common ______ of 12 and — **common factor**

16 30 is ______. — **$2 \cdot 3$, or 6**

We used this observation in completing Item 13.

Similarly, to simplify the fraction $\frac{4x}{2xy}$, we could think:

$$\frac{4x}{2xy} = \frac{2}{y} \cdot \frac{2x}{2x}$$
$$= \frac{2}{y}$$

We followed the same pattern in this example. The expression $2x$ is the greatest common factor of $4x$ and $2xy$.

Notice that if x has the value 3 and y the value 5, then

17 the value of $\frac{4x}{2xy}$ is $\frac{12}{\square}$. — $\frac{12}{30}$

18 the value of $\frac{2}{y}$ is ______. — $\frac{2}{5}$ [or $\frac{12}{30}$]

Indeed, $\frac{4x}{2xy} = \frac{2}{y}$ if x and y are any non-zero real numbers.

The fact that $\frac{2}{y} \cdot \frac{2x}{2x} = \frac{2}{y}$ for all non-zero real

19 numbers follows from the ______ property of 1. — **multiplication**

Applying exactly the same reasoning, we can simplify $\frac{x^2 - 4}{3x - 6}$. (Note that 2 must be excluded as a value of x.)

20	$\frac{x^2 - 4}{3x - 6} = \frac{(\quad)(\quad)}{(\quad)}$ (Factor the numerator and denominator.)	$\frac{(x + 2)(x - 2)}{3(x - 2)}$
21	$= \frac{x + 2}{3} \cdot \frac{\square}{\square}$	$\frac{x - 2}{x - 2}$
	$= \frac{x + 2}{3}$	
22	We see: For all real values of x except _______, $\frac{x^2 - 4}{3x - 6} = \frac{x + 2}{3}$.	2

To simplify $\frac{x^2 - 4}{3x - 6}$, which is an indicated quotient of two polynomials, we wrote it as a quotient of polynomials of as low degree as possible.

	Simplify:	
23	$\frac{3x - 3}{x^2 - 1} =$ _______, provided $x \neq 1$ and $x \neq -1$.	$\frac{3}{x + 1}$
24	$\frac{x^2 - x^2y}{1 - y} =$ _______, provided _______.	x^2; $y \neq 1$
	Notice that x^2 may be considered as the indicated quotient $\frac{x^2}{1}$.	
25	$\frac{x^2 - 4x - 12}{x^2 - 5x - 6} =$ _______, provided _______.	$\frac{x+2}{x+1}$; $x \neq -1$ and $x \neq 6$
26	$\frac{x^4 - 1}{x^3 + x} =$ _______, provided _______.	$\frac{x^2 - 1}{x}$; $x \neq 0$

27	If A, B, C, D are polynomials, $\frac{A}{B} \cdot \frac{C}{D} = \frac{\square}{\square}$. This statement is true, subject to the restriction that we must exclude from the domain those values of the variable for which B is 0 and those for which D is 0.	$\frac{AC}{BD}$

Simplify; that is, write in lowest terms. Assume that the domain of x is properly restricted.

$$\frac{x^2-1}{x}\cdot\frac{2x}{(x+1)^2}=\frac{(x+1)(x-1)}{x}\cdot\frac{2x}{(x+1)(x+1)}$$

$$=\frac{(x+1)(x-1)2x}{x(x+1)(x+1)}$$

28 $=\dfrac{2(x-1)}{x+1}\cdot\dfrac{\square}{\square}$ — $\dfrac{(x+1)x}{(x+1)x}$

29 $=\dfrac{2(x-1)}{x+1}\cdot$ ______ — 1

30 $=$ ______ — $\dfrac{2(x-1)}{x+1}$

Simplify each of the following. Assume that the domain of the variables is properly restricted. If you have difficulty, refer to Items 36-39, where the steps are shown.

31 $\dfrac{x+y}{x-y}\cdot\dfrac{x-y}{x+y}=$ ______ — 1

32 $\dfrac{1-x^2}{1+x}\cdot\dfrac{x-2}{x^2-3x+2}=$ ______ [Hint: $1-x=(-1)(x-1)$] — -1

33 $\dfrac{ab+ab^2}{a-ab^2}\cdot\dfrac{1-b}{1+b}=$ ______ — $\dfrac{b}{1+b}$

34 $\dfrac{ax-bx}{x^2}\cdot\dfrac{a^2+2ab+b^2}{a^2-b^2}=$ ______ — $\dfrac{a+b}{x}$

The following restrictions on the variables in Item 34

35 are necessary. $x \neq$ ______, $a \neq$ ______, and ______. — $x \neq 0$, $a \neq b$ and $a \neq -b$

For Item 31:

$$\frac{x+y}{x-y}\cdot\frac{x-y}{x+y}=\frac{(x+y)(x-y)}{(x-y)(x+y)}$$

36 $=\dfrac{(x+y)}{(x+y)}\cdot\dfrac{\square}{\square}$ — $\dfrac{x-y}{x-y}$

$=1\cdot 1$

37 $=$ ______ — 1

For Item 32:

$$\frac{1-x^2}{1+x} \cdot \frac{x-2}{x^2-3x+2} = \frac{(1-x)(1+x)(x-2)}{(1+x)(x-1)(x-2)}$$

$$= \frac{(-1)(x-1)(1+x)(x-2)}{(x-1)(1+x)(x-2)}$$

$$= -1$$

38 For Item 33:

$$\frac{ab+ab^2}{a-ab^2} \cdot \frac{1-b}{1+b} = \frac{ab(\qquad)(1-b)}{a(\qquad)(\qquad)(1+b)}$$

$$= \frac{b}{1+b}$$

$$\frac{ab(1+b)(1-b)}{a(1-b)(1+b)(1+b)}$$

For Item 34:

39 $$\frac{ax-bx}{x^2} \cdot \frac{a^2+2ab+b^2}{a^2-b^2} = \frac{x(a-b)(a+b)^2}{\boxed{\qquad\qquad}}$$

$$= \frac{a+b}{x}$$

$$\frac{x(a-b)(a+b)^2}{x^2(a-b)(a+b)}$$

We should have no difficulty in simplifying expressions of the form

$$\frac{\frac{A}{B}}{\frac{C}{D}} \quad \text{where } A, B, C, D \text{ are polynomials.}$$

No difficulty, that is, if we properly restrict the variables involved. Notice that it is necessary to exclude all values of the variable for which any one of the polynomials B, C, D is 0.

We can use the following property of real numbers:

40 $$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{\square} \cdot \frac{d}{\square}, \text{ provided } b \neq 0,\ c \neq 0, \text{ and } d \neq 0.$$ $\frac{a}{b} \cdot \frac{d}{c}$

Simplify:

41 $$\frac{\frac{x}{x+1}}{\frac{x-1}{x}} = \frac{x}{x+1} \cdot \frac{x}{\square}$$ $\frac{x}{x+1} \cdot \frac{x}{x-1}$

42 $$= \frac{\square}{(x+1)(x-1)}$$ $\frac{x^2}{(x+1)(x-1)}$

We sometimes prefer $\dfrac{x^2}{x^2 - 1}$ to $\dfrac{x^2}{(x + 1)(x - 1)}$.

It depends on whether the common polynomial form or the factored form of the denominator is to be used later.

43 The set of excluded values of x for the preceding example is ______. $\{-1,0,1\}$

Simplify each of the following: Refer to page ii. if you have difficulty.

44 $\dfrac{\dfrac{x^2 - 9}{6}}{\dfrac{x^2 - 3x}{3x+3}} =$ ______ $\dfrac{(x + 1)(x + 3)}{2x}$

45 $\dfrac{\dfrac{x^2 + x - 2}{x^2 - 4x + 4}}{\dfrac{x + 2}{x - 2}} =$ ______ $\dfrac{x - 1}{x - 2}$

46 $\dfrac{\dfrac{x^2 + 2x + 1}{x^2 - 1}}{\dfrac{x + 1}{x - 1}} =$ ______ 1

Perhaps we had better remind ourselves that it may be necessary to restrict the domain of the variables in indicated quotients of polynomials.

For each of the following, state the set of excluded values of x. You do not need to simplify.

47 $\dfrac{x - \sqrt{2}}{x^2 - 4}$ ______ $\{-2,2\}$

48 $\dfrac{\dfrac{x}{\sqrt{2} - x}}{\dfrac{\sqrt{2} + x}{x}}$ ______ $\{-\sqrt{2},0,\sqrt{2}\}$

49 $\dfrac{\dfrac{x^2 - 5x + 4}{x - 1}}{x - 4}$ ______ $\{1,4\}$

18-4. Rational Expressions

By way of review, simplify:

1 $\frac{5}{6a} + \frac{9}{8a} =$ ______ $\frac{47}{24a}$

2 $\frac{3}{x^2} - \frac{2}{5x} =$ ______ $\frac{15 - 2x}{5x^2}$

3 $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} =$ ______ $\frac{bc + ac + ab}{abc}$

4 $\frac{7}{36a^2b} + \frac{5}{24b^3} =$ ______ $\frac{14b^2 + 15a^2}{72a^2b^3}$

5 $\frac{1}{a^2} - \frac{1}{2a} - 2 =$ ______ $\frac{2 - a - 4a^2}{2a^2}$

$$\frac{5}{x - 1} + 1 = \frac{5}{x - 1} + 1 \cdot \frac{x - 1}{x - 1}$$

$$= \frac{5}{x - 1} + \frac{x - 1}{x - 1}$$

6 $=$ ______ $\frac{x + 4}{x - 1}$

7 In this example we used the ______ property of 1, multiplication

8 writing 1 in the form $\frac{x - 1}{\boxed{}}$. $\frac{x - 1}{x - 1}$

Simplify: $\frac{3}{m - 1} + \frac{2}{m - 2}$

$$\frac{3}{m - 1} + \frac{2}{m - 2} = \frac{3}{m - 1} \cdot \frac{m - 2}{m - 2} + \frac{2}{m - 2} \cdot \frac{m - 1}{m - 1}$$

9 $= \frac{3(m-2) + 2(m-1)}{\boxed{}}$ [Don't forget the parentheses!] $\frac{3(m-2) + 2(m-1)}{(m-1)(m-2)}$

10 $=$ ______ $\frac{5m-8}{(m-1)(m-2)}$

11 $\frac{4}{m - n} + \frac{5}{n} =$ ______ $\frac{5m - n}{n(m-n)}$

12 $\frac{x}{x + 5} - \frac{x}{x + 3} =$ ______ $-\frac{2x}{(x+5)(x+3)}$

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13 $\frac{2}{a - b} - \frac{3}{b - a} =$ ______ [Hint: $b - a = -(\quad)$] — $\frac{5}{a - b}$

14 $\frac{x}{x + y} - \frac{y}{x - y} =$ ______ — $\frac{x^2 - 2xy - y^2}{x^2 - y^2}$

If we encounter indicated quotients with more complicated denominators, we proceed as before, using our knowledge of factoring.

$$\frac{a}{3a - 9} - \frac{2a - 3}{5a - 15} = \frac{a}{3(a - 3)} - \frac{2a - 3}{5(a - 3)}$$

$$= \frac{a}{3(a - 3)} \cdot \frac{5}{5} - \frac{2a - 3}{5(a - 3)} \cdot \frac{3}{3}$$

15 $= \frac{5a - (2a - 3)(3)}{\boxed{\quad\quad}}$ [Don't forget!] — $15(a - 3)$

16 $=$ ______ — $\frac{-a + 9}{15(a - 3)}$

17 $\frac{5x}{x^2 - 9} + \frac{7}{x + 3} = \frac{5x}{(x + 3)(x - 3)} + \frac{7}{x + 3} \cdot \frac{\square}{\square}$ — $\frac{x - 3}{x - 3}$

$$= \frac{5x + 7(x - 3)}{(x + 3)(x - 3)}$$

18 $=$ ______ — $\frac{12x - 21}{(x+3)(x-3)}$

19 $\frac{7}{a - b} + \frac{6}{a^2 - 2ab + b^2} =$ ______ — $\frac{7a - 7b + 6}{(a - b)^2}$

20 $\frac{3}{x^2 + 2x} - \frac{5}{3x + 6} =$ ______ — $\frac{9 - 5x}{3x(x+2)}$

21 $\frac{4}{a^2 - 4a - 5} + \frac{2}{a^2 + a} =$ ______ — $\frac{6a - 10}{a(a-5)(a+1)}$

Here are some more, for practice. Simplify:

22 $\frac{5}{x^2 + x - 6} + \frac{3}{x^2 - 4x + 4} =$ ______ — $\frac{8x - 1}{(x-2)^2(x+3)}$

23 $\frac{a}{3 + a} - \frac{a - 3}{a} =$ ______ — $\frac{9}{a(a + 3)}$

24 $\frac{y - 5}{2y} + \frac{y + 5}{y^2} =$ ______ — $\frac{y^2 - 3y + 10}{2y^2}$

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25 $\frac{4}{x^2 - x} + \frac{3}{x - 1} - \frac{1}{x} =$ ______ $\frac{2x + 5}{x(x - 1)}$

26 $\frac{a}{a^2 - 25} - \frac{2}{3a + 15} + \frac{3}{2a - 10} =$ ______ $\frac{11a + 65}{6(a + 5)(a - 5)}$

Consider the phrase

$$\frac{x - \frac{y^2}{x}}{1 + \frac{y}{x}}.$$

We may simplify this phrase as follows:

$$\frac{x - \frac{y^2}{x}}{1 + \frac{y}{x}} \cdot \frac{x}{x} = \frac{x^2 - y^2}{x + y}$$

$$= \frac{(x + y)(x - y)}{x + y}$$

$$= x - y$$

In simplifying $\frac{x - \frac{y^2}{x}}{1 + \frac{y}{x}}$, you might prefer to begin by writing the numerator and the denominator each as a single fraction.

27 $x - \frac{y^2}{x} = \frac{\square}{x}$ $\frac{x^2 - y^2}{x}$

28 $1 + \frac{y}{x} = \frac{\square}{\square}$ $\frac{x + y}{x}$

Hence, $\frac{x - \frac{y^2}{x}}{1 + \frac{y}{x}} = \frac{\frac{x^2 - y^2}{x}}{\frac{x + y}{x}}$

$$= \frac{x^2 - y^2}{x} \cdot \frac{x}{x + y}$$

29 $=$ ______ $x - y$

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Simplify:

30 $\dfrac{\frac{1}{a} + \frac{1}{b}}{\frac{1}{a^2} - \frac{1}{b^2}} =$ ______ [Hint: Multiply by $\dfrac{a^2b^2}{a^2b^2}$.] $\dfrac{ab}{b - a}$

31 $\dfrac{\frac{x}{3} - 2 + \frac{3}{x}}{1 - \frac{3}{x}} =$ ______ $\dfrac{x - 3}{3}$

32 $(1 - \frac{1}{x + 1})(1 + \frac{1}{x - 1}) =$ ______ [Hint: Add within parentheses first.] $\dfrac{x^2}{x^2 - 1}$

In this and the preceding sections we have worked with expressions such as:

$$\frac{5x}{x^2 - 9} + \frac{7}{x + 3} \qquad \text{(Item 17-18)}$$

$$\frac{\frac{x}{3} - 2 + \frac{3}{x}}{1 - \frac{3}{x}} \qquad \text{(Item 31)}$$

$$(1 - \frac{1}{x + 1})(1 + \frac{1}{x - 1}) \qquad \text{(Item 32)}$$

Such phrases are called rational expressions.

Definition. A phrase formed from members of a set consisting of the real numbers and one or more variables and using at most the operations of addition, subtraction, multiplication, and division is called a rational expression.

If you refer to the items noted above, you will note that every rational expression in one variable can be expressed as the quotient of two polynomials in common polynomial form.

Again we are able to observe a similarity with our earlier experience. We recall that every rational number can be expressed as the quotient of two integers.

Although $\dfrac{5 - 3(\frac{1}{2})}{\frac{3}{4}}$ is not written as the quotient of two integers, it is a rational number, since it can be written (as you should verify) as $\frac{14}{3}$.

33	$\dfrac{\frac{x}{2} - \frac{3}{x - 1}}{2 - \frac{x}{x + 1}}$ is a ____ expression in one variable, according to our definition.	rational expression
34	We may write it as $\dfrac{}{2(x - 1)}$	$\dfrac{(x - 3)(x + 1)}{2(x - 1)}$
35	We observe that $\dfrac{(x - 3)(x + 1)}{2(x - 1)}$ is the indicated ____ of two polynomials. We might write it as $\dfrac{x^2 - 2x - 3}{2x - 2}$, which is the quotient of two polynomials in common polynomial form.	quotient

Notice that the expression $x^2 - 3x + 2$ fits our definition of rational expression. We can write it, if we like, as the quotient of two polynomials:

$$x^2 - 3x + 2 = \frac{x^2 - 3x + 2}{1}.$$

36 Which of the following statements is false?

[A] Every polynomial is a rational expression.

[B] Every rational expression may be written as the indicated quotient of two polynomials.

[C] Every rational expression may be written as a polynomial.

[D] $\frac{1}{x}$, 2, $\frac{1}{x} + 2$, $x(\frac{1}{x} + 2)$ and $\dfrac{x(\frac{1}{x} + 2)}{x - 1}$ are all rational expressions.

The definition of rational expression tells us that [A] and [D] are true statements. Our whole development in Sections 18-3 and 18-4 indicates that [B] is true. [C] is false, since, for example, $\frac{1}{x}$ is a rational expression but $\frac{1}{x}$ cannot be written as a polynomial.

For each of the following, respond NRE if the phrase is not a rational expression. If the phrase is a rational expression, write it as the indicated quotient of two polynomials having no common factor.

37	$\frac{1}{x} + 1$	$\dfrac{x + 1}{x}$
38	$\dfrac{1}{\lvert x - 3 \rvert} - 7$	NRE

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39	$\frac{z}{z-1} \cdot \frac{z-1}{5}$	$\frac{z}{5}$
40	$\dfrac{x - \frac{1}{x}}{x^2 - 1}$	$\frac{1}{x}$
41	$\dfrac{x - \frac{1}{\sqrt{x}}}{x - 1}$	[illegible] NRE
42	$\dfrac{\frac{x}{2} - \frac{3}{x-1}}{\frac{x}{2} - \frac{1}{x+1}}$	[See page 11.]

Summary and Review

In this chapter we have considered quotients of polynomials.

We have seen that if N and D are polynomials in one variable, with D different from 0, then there exist polynomials Q and R such that

$$N = QD + R,$$

where either R is 0 or R has degree lower than that of D.

We may restate this result as:

$$\frac{N}{D} = Q + \frac{R}{D}.$$

We defined rational expressions, noting that their relationship to polynomials resembles that of the rational numbers to the integers.

Review Problems

1. Simplify the following rational expressions:

(a) $\dfrac{\dfrac{3x^2y^6}{20a^2b^2}}{\dfrac{7(xy^2)^3}{30(ab^2)^2}}$

(b) $\frac{3}{35a^2} + \frac{13}{25ab} - \frac{5}{7b^2}$

(c) $\frac{2}{a^2 - ab} + \frac{3}{b^2 - ab} + \frac{4}{ab}$

(d) $\frac{x}{x^2 - 9} + \frac{2x - 5}{x^2 - 4x + 3} - \frac{3x}{x^2 + 2x - 3}$

2?0

2. Divide the given polynomials.

(a) $\dfrac{x^3 - 4x^2 + x + 6}{x - 3}$ (c) $\dfrac{x^3 - 1}{x + 1}$

(b) $\dfrac{3x^4 + 14x^3 - 4x^2 - 11x - 2}{3x + 2}$ (d) $\dfrac{x^5 - 1}{x - 1}$

3. Simplify $1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{2}}}$.

4. A rug with area of 24 square yards is placed in a room 14 feet by 20 feet, leaving a uniform width around the rug. How wide is the strip around the rug? [Hint: Draw a sketch.]

5. Factor:

(a) $x^2 - 22x - 48$

(b) $x^2 - y^2 - 4x - 4y$

(c) $3a^3b^5 - 6a^2b^3 + 12a^4b^4$

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Chapter 19

TRUTH SETS OF OPEN SENTENCES

Throughout this course we have practiced solving open sentences. Our general procedure, which we emphasized in Section 9-3, is to create a chain of equivalent open sentences, finally obtaining an open sentence whose truth set (solution set) is obvious. This method, you will recall, depends on the idea that the steps taken in deriving one sentence from another are <u>reversible</u> steps. In this chapter we will examine carefully the question of which algebraic operations on sentences are "permissible"; that is, which operations lead from one open sentence to an equivalent one. You may wish to review briefly the discussion in Section 9-3 before continuing.

19-1. <u>Equivalent Equations</u>

1 Which of the following <u>always</u> leads from one equation to an equivalent equation?

I. Adding the same real number to both sides.

II. Multiplying both sides by the same real numbers.

[A] I only [B] II only [C] both [D] neither

The correct choice is [A]. A discussion of this question is the main topic of this section.

	$x - 7 = 5$ is equivalent to $x = 5 + 7$	
2	$= \_\_\_\_\_$.	12
3	We obtained $x = 12$ by adding _____ to both sides of the original equation.	7
4	The step is reversible. If we start with $x = 12$, we may obtain the original equation by subtracting _____ from both sides.	7
	Remember that "subtract 7" means the same as	
5	"add the _____ of 7", or	opposite
6	"add the additive _____ of 7".	inverse

If we start with any equation and add a certain real number to both sides to obtain a second equation, we may reverse this step by subtracting that same real number from both sides of the second equation. Our justification for this reasoning is based on the fact that every real number has exactly one additive inverse.

7	Which of the following pairs of equations are equivalent? [A] $x + 2 = -5,\ x = -3$ [B] $5s + 1 = 4 + 4s,\ s = 3$ [C] $6 - t = 7,\ t = 1$	

[B] is the correct choice. The solution set of the first sentence of [A] is $\{-7\}$. The solution set of the first sentence of [C] is $\{-1\}$.

8	Adding the same real number to both sides of an equation is a permissible operation since every real number has exactly one ______ inverse.	additive
9	Does every real number have exactly one multiplicative inverse? ______ (yes, no)	no
10	Every real number except ______ has a unique multiplicative inverse.	0
11	Another name for multiplicative ______ is reciprocal.	inverse
12	$\frac{1}{3}x = 7$ is equivalent to $x = 7 \cdot 3$ $= $ ______.	21
13	We obtained $x = 21$ by multiplying both sides of the original equation by ______.	3
14	The step is reversible. If we start with $x = 21$, we may obtain the original equation by dividing both sides by ______.	3
15	Remember that "divide by 3" means the same as "multiply by the ______ inverse of 3" or as	multiplicative
16	"multiply by the reciprocal of ______."	3

If we start with any equation, and multiply both sides by a certain <u>non-zero</u> real number to obtain a second equation, we may reverse this step by dividing both sides of the second equation by that same non-zero real number. Our justification for this reasoning is based on the fact that every non-zero real number has exactly one multiplicative inverse.

Solve each of the following equations. Answers are on page iv.

17. $2s = 12$	23. $3x^2 - 6x = 0$
18. $5s = 3s + 12$	24. $\frac{1}{7}x = \frac{1}{105}$
19. $5y - 4 = 3y + 8$	25. $\frac{5}{8}x - 17 = 33$
20. $3x + 9 - 2x = 7x - 12$	26. $y^4 + y^3 + y^2 + y + 1 = y^4 + y^3 + y^2 + 1$
21. $4 - 2x = 10$	27. $x^2 - 5x + 6 = 0$
22. $x^2 + 5 = 1$	28. $x(x + 1) = x^2 + x$

Suppose we wish to multiply or divide both sides of an equation by a phrase that contains a variable. In order to solve

$$x(x^2 + 1) = 2(x^2 + 1)$$

we are tempted to divide both sides by $x^2 + 1$. Is the resulting equation equivalent to the original one?

29	The truth set of $x(x^2 + 1) = 2(x^2 + 1)$ is ______.	{2}
30	The truth set of $x = 2$ is ______; hence, the	{2}
31	sentences are ______.	equivalent
	Notice that whatever real number x represents, $x^2 + 1$ names a non-zero real number!	

	To solve $x(x - 3) = 2(x - 3)$ we are tempted to divide	
32	both sides by $x - 3$, and obtain $x =$ ______.	$x = 2$
33	The solution set of $x = 2$ is ______.	{2}
34	On the other hand, 3 ______ (is, is not) a solution of $x(x - 3) = 2(x - 3)$.	is

35	$x = 2$ is not ______ to $x(x - 3) = 2(x - 3)$.	equivalent
36	Notice that if x is 3, then $x - 3$ has the value ______, and we may not divide by 0. We will return to this equation shortly.	0

37 May we multiply both sides of

$$\frac{x^2}{x^2 + 1} = \frac{1}{2} \text{ by } 2(x^2 + 1)$$

and obtain an equivalent equation?

[A] yes [B] no

For every value of the variable, $2(x^2 + 1)$ names a non-zero real number. The proper choice is [A].

38	Solve $\dfrac{x^2}{x^2 + 1} = \dfrac{1}{2}$.	$\{-1, 1\}$
39	Solve $\dfrac{x^2 + 5}{x^2 + 5} = 0$.	$\emptyset$
40	Solve $\dfrac{x^2 + 5}{x^2 + 5} = 1$.	set of all real numbers
	If you had difficulty with Items 38-40, see page v.	

41	Solve $3x(2x^2 + 3) = 5(2x^2 + 3)$.	$\{\frac{5}{3}\}$
42	Solve $x^2(\lvert x\rvert + 1) = 4(\lvert x\rvert + 1)$.	$\{-2, 2\}$
43	Solve $x(3x^2 + 4) = (3x^2 + 4)$.	$\{1\}$

In the last few items we have considered cases where we multiplied or divided both sides of an equation by an open phrase which named a non-zero real number for every value of the variable. Let us return to the open sentence $x(x - 3) = 2(x - 3)$.

If we divide both sides of

$$x(x - 3) = 2(x - 3) \text{ by } x - 3$$

44 we ______ obtain an equivalent equation. — do not
(do, do not)

	How then do we solve $x(x - 3) = 2(x - 3)$?	
45	$2(x - 3)$ _______ (does, does not) name a real number for every value of the variable.	does
	Hence, we may subtract $2(x - 3)$ from both sides and	
46	obtain an _______ equation.	equivalent
	$x(x - 3) = 2(x - 3)$ is equivalent to:	
47	$x(x - 3) - 2(x - 3) =$ _______	0
48	$(\quad)(x - 3) =$ _______	$(x - 2)(x - 3) = 0$
49	$x - 2 = 0$ or _______	$x - 2 = 0$ or $x - 3 = 0$
	Therefore, the solution set of $x(x - 3) = 2(x - 3)$	
50	is _______.	$\{2,3\}$

We have discovered that

$$x(x - 3) = 2(x - 3) \text{ is equivalent to}$$

$$x - 2 = 0 \text{ or } x - 3 = 0.$$

The solutions of this compound sentence are 2 and 3.

	$2x(3x - 1) = 5(3x - 1)$ is equivalent to the compound sentence	
51	$2x - 5 = 0$ or _______.	$3x - 1 = 0$
52	There are two solutions, $\frac{5}{2}$ and _______.	$\frac{1}{3}$
	$x^2 = \frac{1}{2}x$ is equivalent to the compound sentence	
53	$x - \frac{1}{2} =$ ___ or _______.	$x - \frac{1}{2} = 0$ or $x = 0$
54	The solution set is _______.	$\{\frac{1}{2}, 0\}$
	$x(x^2 + 1) = 2(x^2 + 1)$ is equivalent to	
55	___ $= 0$ or ___ $= 0$	$x - 2 = 0$ or $x^2 + 1 = 0$
56	The equation has _______ (how many) solutions?	one

57 If a, b, c are real numbers and if we know that $ac = bc$, then we may conclude that

[A] $a = b$ [C] $a = b$ or $c = 0$

[B] $c = 0$ [D] $a = b$ and $c = 0$

If $ac = bc$, we may subtract bc from both sides. Hence, $ac - bc = 0$. Factoring, we have $(a - b)c = 0$. Therefore, $a - b = 0$ or $c = 0$. The correct choice is [C].

$(3x + 1)(x - 2) = (x - 5)(x - 2)$ is equivalent to the compound sentences:

58 $3x + 1 = x - 5$ or $\_\_\_\_ = 0$ — $x - 2 = 0$

$2x = -6$ or $x = 2$

59 $x = \_\_\_\_$ or $\_\_\_\_$ — $x = -3$ or $x = 2$

$x(2x - 5) = 7x$ is equivalent to

60 $2x - 5 = \_\_\_\_$ or $x = 0$ — $2x - 5 = 7$

61 $2x = 12$ or $\_\_\_\_$ — $x = 0$

62 $x = \_\_\_\_$ — $x = 6$ or $x = 0$

$(x - 3)(x^2 - 1) = 4(x^2 - 1)$ is equivalent to

63 $\_\_\_\_$ or $x^2 - 1 = 0$ — $x - 3 = 4$

64 $x = \_\_\_\_$ or $x = \_\_\_\_$ or $x = \_\_\_\_$. — $x = 7$ or $x = -1$ or $x = 1$

$3(x^2 - 4) = (4x + 3)(x^2 - 4)$ is equivalent to

65 $x = \_\_\_\_$ or $x = \_\_\_\_$ or $x = \_\_\_\_$. — $x = 0$ or $x = -2$ or $x = 2$

We know that if a and b are real numbers, then $ab = 0$ if and only if $a = 0$ or $b = 0$. We have been using this notion extensively. The result may be extended to more than two factors. For example,

$$\begin{aligned} abcd = 0 \quad &\text{if and only if} \quad abc = 0 \text{ or } d = 0 \\ &\text{if and only if} \quad ab = 0 \text{ or } c = 0 \text{ or } d = 0 \\ &\text{if and only if} \quad a = 0 \text{ or } b = 0 \text{ or } c = 0 \text{ or } d = 0. \end{aligned}$$

66	$x(x - 1)(x - 2)(x - 3) = 0$ has the solution set ______.	$\{0,1,2,3\}$
67	$(3x^2 + 4)(x^2 - 1)(x^2 - 7x + 12) = 0$ is equivalent to $(3x^2 + 4)(x + 1)(x - 1)(\quad)(\quad) = 0$.	$(x - 3)(x - 4)$.
68	The solution set of $3x^2 + 4 = 0$ is ______.	$\emptyset$
69	The solutions of $(3x^2 + 4)(x^2 - 1)(x^2 - 7x + 12) = 0$ are -1, ___, 3, and ___.	-1, 1, 3, and 4
70	Solve $x^3 - 9x = 0$. The truth set is ______.	$\{-3,0,3\}$
71	The solutions of $(3x - 1)(2x + 1)(4x - 3) = 0$ are ___, ___, and ___.	$\frac{1}{3}$, $-\frac{1}{2}$, and $\frac{3}{4}$
72	$x^2 = 121$ is equivalent to $x =$ ___ or ___.	$x = -11$ or $x = 11$
73	$-4(\lvert 3x \rvert + 1)(x + 2)$[illegible]$ = 0$ is equivalent to $x =$ ___.	$x = -2$

Perhaps you have noticed in the preceding items that we often asked for a response which simply listed the members of a solution set or for a response which consisted of an equivalent sentence which has an obvious truth set. This is in accord with the practice of many mathematicians. Instead of saying

$$\text{the solution set of } 3x - 4 = 17 \text{ is } \{7\},$$

we might say,

$$\text{the solution of } 3x - 4 = 17 \text{ is } 7,$$

or,

$$\text{if } 3x - 4 = 17, \text{ then } x = 7.$$

We shall continue to be careful to word our items so that you will know what response to give. In particular, we shall continue to use "solve" to mean "find the solution set".

Here are some more problems to provide practice in solving equations. If you feel that you do not need more practice, omit Items 74-83. For each equation, find the solution set. Answers are on pages v and vi.

74. $(x^2 - 5)(2x - 1) = 0$

75. $(x^2 - 7)(x^2 - 24) = 0$

76. $y^3 = 25y$

77. $x^3 + x = 2x^2$

78. $x^2 + 6x + 1 = 0$

79. $6x^2(x - 1)(2x + 1) = 0$

80. $(t^2 - 4)^3 = 0$

81. $x^2(x - 1) = 3(x - 1)$

82. $x^2(x^4 + 2) = 9(x^4 + 2)$

83. $\frac{x}{2}(3x - 1) = \frac{1}{5}(15x - 5)$

In this section, we have not "checked" our obtained solutions in the original equation. If we proceed from one sentence to another using only permissible operations, we may be sure that the sentences are equivalent. Checking, however, does help to reveal whether careless errors have been made.

Let us see whether $-3 + 2\sqrt{2}$ is a solution of

$$x^2 + 6x + 1 = 0.$$

If $-3 + 2\sqrt{2}$ is a solution, then

$$(-3 + 2\sqrt{2})^2 + 6(-3 + 2\sqrt{2}) + 1 = 0$$

84 is a ______ (true, false) sentence. — true

85 $(-3 + 2\sqrt{2})^2 = 9$ ______ — $9 - 12\sqrt{2} + 8$

$= 17 - 12\sqrt{2}$

86 $6(-3 + 2\sqrt{2}) =$ ______ — $-18 + 12\sqrt{12}$

87 $(17 - 12\sqrt{2}) + (-18 + 12\sqrt{12}) + 1$ ______ $(=, \neq)$ 0. — $=$

88 Which of the following always leads from one equation to an equivalent equation?

I. Adding the same real number to both sides.

II. Multiplying both sides by the same real number.

[A] I only [B] II only [C] both [D] neither

This is a repeat of Item 1 of this section. The correct choice is [A]. Multiplying by a <u>non-zero</u> real number also leads to an equivalent equation.

89 Which of the following polynomials has the value 0 for <u>all</u> values of x in the set $\{2,-1,0\}$?

[A] $(x + 2)(x - 2)(x)$
[B] $(x - 2)(x + 1)(x)$
[C] $(x - 2)(x + 1)$

The correct choice is [B]. We know that $(x - 2)(x + 1)(x)$ will be zero if and only if $x - 2 = 0$ or $x + 1 = 0$ or $x = 0$. So we see that $(x - 2)(x + 1)(x) = 0$ if and only if x takes a value from the set $\{2,-1,0\}$.

Do you see a method for finding a polynomial with the value 0 whenever x takes a value in a given set? For instance, consider the set $\{a,b,c,d\}$. A polynomial which is 0 whenever x is a number in this set is $(x - a)(x - b)(x - c)(x - d)$. This leads, in turn, to a method for writing equations if we have the truth set.

If the truth set of an open sentence is $\{3,-2\}$, then

90 $x = 3$ or $x =$ ______, which is equivalent to — -2

91 $(\quad)(\quad) = 0$, which is equivalent to — $(x - 3)(x + 2) = 0$

92 ________ $= 0$. — $x^2 - x - 6 = 0$

Let us write an open sentence having the truth set $\{0,1,-1\}$.

Some possible sentences are:

93 $x = 0$ or $x = 1$ or ______ — $x = -1$

94 $x(\quad)(\quad) = 0$, — $x(x - 1)(x + 1) = 0$

95 x^3 ____ $= 0$. — $x^3 - x = 0$

*96. Write a polynomial of degree three having integers as coefficients which has the value 0, for each of the following values of the variable: $\frac{2}{3}, -\frac{1}{2}, \frac{5}{6}$. Answer is on page vi.

19-2. Equations Involving Fractions

Our work with factoring polynomials led naturally to the solution of polynomial equations. We turn now to solving equations involving rational expressions. Such equations arise in a great variety of mathematical applications. Some examples are given at the end of this section.

First of all, if an equation involves fractions having real numbers as denominators, we should have no difficulty. Let us solve such an equation by two different methods in order to prepare us for more complicated problems.

	To solve $\frac{4}{3} - \frac{y}{5} = \frac{1}{2}$, we might first multiply both sides of the equation by the least common multiple of	
1	the denominators; in this case, by ______.	30
2	We obtain the equivalent equation ______ $- 6y =$ ______.	$40 - 6y = 15$
3	The truth set is ______.	$\{\frac{25}{6}\}$

	Another method of solving $\frac{4}{3} - \frac{y}{5} = \frac{1}{2}$ would be to first subtract $\frac{1}{2}$ from both sides and proceed as follows:	
4	$\frac{4}{3} - \frac{1}{2} - \frac{y}{5} =$ ______. Writing the left hand side as a single fraction we have:	0
5	$\frac{40 - 15 - 6y}{\square} = 0$, or $\frac{25 - 6y}{30} = 0$.	30
6	Hence, $25 - 6y =$ ______ is equivalent to the original	$25 - 6y = 0$
7	equation, and the truth set is ______.	$\{\frac{25}{6}\}$

8 Our second method (Items 4-7) depended on the following:

If a and b are real numbers and $\frac{a}{b} = 0$, then we know that:

[A] $a = 0$
[B] $a = 0$ and $b = 0$
[C] $a = 0$ or $b = 0$
[D] $a = 0$ and $b \neq 0$

If $b = 0$, then $\frac{a}{b}$ is not a real number. The correct choice is [D].

It turns out, in actually solving equations involving rational expressions that it is usually easier to use the first of our two methods--that of multiplying by the least common multiple of the denominators. If the denominators involve variables, however, our work in Section 19-1 should warn us that we need to be careful.

9	Is $\frac{x^2}{x} = 1$ equivalent to $x^2 = x$? [A] yes [B] no	

0 is a solution of $x^2 = x$, but if x is 0, then $\frac{x^2}{x}$ does not name a real number. [B] is correct.

	We observe that the following sentences are all equivalent. $\frac{1}{x} = 2.$	
10	$\frac{1}{x} = 2$ and x ______.	$x \neq 0$
11	$\frac{1}{x} \cdot x =$ ______ and $x \neq 0$.	$2x$
	$1 = 2x$ and $x \neq 0$.	
12	The solution set of $\frac{1}{x} = 2$ is ______.	$\{\frac{1}{2}\}$

13	To solve $\frac{1}{x} = \frac{2}{1 - x}$ we may multiply both sides of the equation by $x(1 - x)$, obtaining $1 - x =$ ______, remembering that certain values of x are not permissible.	$1 - x = 2x$
14	Thus, the sentence $\frac{1}{x} = \frac{2}{1 - x}$ is equivalent to "$1 - x = 2x$ and $x \neq$ ___ and $x \neq$ ___".	$x \neq 0$ and $x \neq 1$
15	The solution of $\frac{1}{x} = \frac{2}{1 - x}$ is ______.	$\frac{1}{3}$
	We could have obtained this result by constructing the chain of equivalent sentences: $\frac{1}{x} = \frac{2}{1 - x}$	
16	$\frac{1}{x} - \frac{2}{1 - x} =$ ______	0
17	$\frac{\square}{\square} = 0$	$\frac{1 - x - 2x}{x(1 - x)}$ or $\frac{1 - 3x}{x(1 - x)}$
	$1 - 3x = 0$ and $x \neq 0$ and $x \neq 1$.	

Solve $\frac{3}{2y} - \frac{[illegible]}{y} = \frac{1}{3}$ [illegible]

To obtain an equation without fractions we multiply

[illegible] by ______. 6y

The given sentence is equivalent to

[illegible] ______. $y \neq 0$

20 The solution set is ______. $\{-\frac{3}{32}\}$

[illegible] first multiplying both members of the equation [illegible]. Think through each step.

The solution set is ______. {1}

If you missed it, or if you are not sure you understand the process, see page [illegible].

Solve $\frac{x}{x - 2} = \frac{2}{x - 2}$.

The sentence $\frac{x}{x - 2} = \frac{2}{x - 2}$ is equivalent to

"x - 2 = 0 and x ≠ ______". $x \neq 2$

The truth set of this compound sentence is ______. $\emptyset$

To solve

$$\frac{-2}{x - 2} + \frac{x}{x - 2} = 1,$$

we may simplify the left member to obtain:

$$\frac{\square}{\square} = 1.$$ $\frac{x - 2}{x - 2}$

The solution set is the set of ______. **all real numbers except 2**

Which of the following sentences have truth sets with an element in common?

R. $\frac{2}{x} - \frac{3}{x} = 10$ T. $y - \frac{2}{y} = 1$

S. $\frac{x}{2} - \frac{x}{3} = 10$ U. $\frac{1}{y} - \frac{1}{y - 4} = 1$

[A] R and S [B] R and U [C] T and U

The truth sets are as follows:

R. $\{-\frac{1}{10}\}$ **S.** $\{60\}$ **T.** $\{-1,2\}$ **U.** $\{2\}$

2 is an element of each of the last two sets, so [C] is correct.

27 Solve each of the following equations. Then decide which of them have $\emptyset$ for the truth set.

R. $\frac{s-2}{s} + \frac{3}{s^2} = 1$ T. $\frac{1}{t} = \frac{1}{t-1}$

S. $\frac{1-y}{1+y} + \frac{1+y}{1-y} = 0$ U. $\frac{1-y}{1+y} - \frac{1+y}{1-y} = 0$

[A] S and T [C] S, T, and U

[B] none of them [D] all of them

[A] is correct. Did you get $\{\frac{3}{2}\}$ for the truth set of R and $\{0\}$ for the truth set of U ? Compare the solutions of the four equations given on page vi with your own.

Solve: $\left(\frac{x-1}{x+1}\right)^2 = 4$. If you have difficulty, see page vi.

28 The solution set is ______. **$\{-3,-\frac{1}{3}\}$**

29 The equation $\frac{x^2-1}{x-1} = 0$ has ______ (how many) solutions. **one**

30 This solution is ______. **-1**

31 $\frac{x^2}{x} = 4$ has ______ (how many) solutions? **one**

32 $x^2 = 4x$ has ______ (how many) solutions? **two**

33 " $\frac{x^2}{x} = 4$ " and " $x^2 = 4x$ " ______ (are, are not) equivalent equations. **are not**

An open sentence which is equivalent to $\frac{x^2}{x} = 4$ is

34 " $x^2 = 4x$ and ______." **$x \neq 0$**

	To solve $\frac{x(x^2 - 1)}{x + 1} = 0$, we notice that this is	
[illegible]	equivalent to $x(x^2 - 1) = 0$ and $x + 1 \neq$ ______.	$x + 1 \neq 0$
	$x(x^2 - 1) = 0$ has three solutions: namely, -1, 0, and 1.	
[illegible]	The solution set of $\frac{x(x^2 - 1)}{x - 1} = 0$ is ______.	$\{0, 1\}$

We might explain our work in this section in the following way: When solving an equation involving rational fractions, we must restrict the domain of the variable so that no denominator takes on the value 0. You will notice the similarity of this restriction to those discussed in Chapter 13 on fractions and in Chapter 18 on rational expressions.

	In solving,	the domain of x is all real numbers except	
37	$\frac{1}{x} + \frac{2}{x}$	______	0
38	$\frac{x^2 - 4}{x(x - 2)} = 3$	______	$0,\ 2$
39	$\frac{1}{(x + 1)(x - 2)} - \frac{2}{x(x - 2)} = 0$	______	$-1,\ 0,\ 2$
40	$\frac{2}{3x - 1} + \frac{x}{2x + 1} = 1$	______	$\frac{1}{3},\ -\frac{1}{2}$

* In solving $\frac{x}{24} - \frac{3}{20} = \frac{1}{15}$, we would use our knowledge about factoring integers to choose the multiplier $24 \cdot 5 = 120$ rather than the multiplier $24 \cdot 20 \cdot 15 = 720$. In the same way we may use our knowledge of factoring polynomials in solving

$$\frac{1}{x^2 - 3x + 2} + \frac{3}{x^2 - x} = \frac{2}{x^2 - 2x}$$

	Factoring the denominators of $\frac{1}{x^2 - 3x + 2} + \frac{3}{x^2 - x} = \frac{2}{x^2 - 2x}$ we have	
*41	$\frac{1}{\_\_\_\_} + \frac{3}{x(x-1)} = \frac{2}{x(x-2)}$	$(x - 1)(x - 2)$
*42	We multiply both sides by ______.	$x(x - 1)(x - 2)$

	Our original equation is equivalent to:	
*43	$x + 3(x - 2) = 2$ ____ and $x \neq 0,\ x \neq 1,\ x \neq 2.$	$2(x - 1)$
*44	The solution set is ____.	$\emptyset$
	Solving $x + 3(x - 2) = 2(x - 1)$ leads to $x = 2$, but 2 is one of our excluded values of x.	

	Solve each of the following:	
*45	$1 + \frac{12}{x^2 - 4} = \frac{3}{x - 2}$	$\{1\}$
*46	$\frac{1}{x^2} + \frac{1}{x} = 2$	$\{-\frac{1}{2}, 1\}$
*47	$\frac{x}{x - 1} + \frac{1}{x^2 - 1} + \frac{x}{x + 1} = 3$	$\{-2, 2\}$

It would be good practice if you were to check the solutions in the original equations in Items *45-*47.

Here are some problems that lead to equations involving fractions. If your answer is not correct, or if you are not sure of how to proceed, complete the items below the problem.

	The sum of two numbers is 8 and the sum of their reciprocals is $\frac{2}{3}$.	
48	What are the numbers? ____	6 and 2

49	One of two numbers whose sum is 8 may be represented by $8 - x$ if the other number is ____.	x
50	An appropriate open sentence is ____ $+ \frac{1}{8 - x} =$ ____	$\frac{1}{x} + \frac{1}{8 - x} = \frac{2}{3}$
51	In solving the sentence $\frac{1}{x} + \frac{1}{8 - x} = \frac{2}{3}$, we note the restrictions $x \neq$ ____ and $x \neq$ ____.	$x \neq 0$ and $x \neq 8$
	To solve $\frac{1}{x} + \frac{1}{8 - x} = \frac{2}{3}$, we will multiply by $3x(8 - x)$.	
52	We obtain: $3(8 - x) +$ ____ $=$ ____ $(8 - x)$.	$3(8-x)+3x=2x(8-x)$

The open sentence $3(8 - x) + 3x = 2x(8 - x)$ is

53 is equivalent to $2x^2 -$ ______ $x + 24 = 0$. — 16

54 Hence, $x^2 - 8x +$ ______ $= 0$, — 12

or $(x - 6)(x - 2) = 0$.

This is equivalent to $x - 6 = 0$ or $x - 2 = 0$

55 and, finally, to "$x =$ ______ or $x =$ ______." — $x = 6$ or $x = 2$

The truth set is $\{6,2\}$.

We can see: If one number is 6, the other is 2,

56 their sum is ______ and the sum of their — 8

57 reciprocals, $\frac{1}{6} + \frac{1}{2}$, is ______. — $\frac{2}{3}$

In a certain school the ratio of boys to girls was $\frac{7}{6}$.

If there were 2600 students in the school, how many

58 girls were there? ______ — 1200

If g represents the number of girls in the

59 school, then there were ______ boys. — $2600 - g$

Since the ratio of boys to girls is $\frac{7}{6}$ we write

60 $\frac{2600 - g}{g} =$ ______ and $g \neq 0$. — $\frac{7}{6}$

We must find the solution set of the compound sentence: $\frac{2600 - g}{g} = \frac{7}{6}$ and $0 < g < 2600$, where g is an integer.

To find the solution set of $\frac{2600 - g}{g} = \frac{7}{6}$, we must find an equivalent sentence whose solution set is obvious.

The first step, noting that $g \neq 0$, is to write

61 $6(2600 - g) =$ ______. — $7g$

For the sentence $6(2600 - g) = 7g$, an equivalent

62 sentence is ______ $= 7g$. — $15600 - 6g$

63 The truth set of the equation is ______. — $\{1200\}$

64 Therefore, there are ______ girls and ______ boys in the school. Note that $\frac{1400}{1200} = \frac{7}{6}$. — 1200 girls, 1400 boys

A mixture for killing weeds must be made in the ratio of 3 parts weed killer to 17 parts water.
How many quarts of weed killer should be put in a 10 gallon tank which is going to be filled up with water to make 10 gallons of mixture? _______ | 6 quarts

(Hint: 10 gallons = _______ quarts.) | 4

If k quarts of weed killer are used, then _______ quarts of water will be needed to fill the tank. Notice that $k + (40 - k) = 40$. | $40 - k$

This means that $\frac{k}{\square} = \frac{3}{17}$. | $\frac{k}{40 - k}$

Solving the sentence $\frac{k}{40 - k} = \frac{3}{17}$ we find the truth set _______. | {6}

70 There should be _______ quarts of weed killer used. | 6

71 So we would need _______ quarts of water. | 34

Notice that $\frac{6}{34} = \frac{3}{17}$.

72 A printing company has three presses: A, B, and C. Press A can do a certain job in 3 hours, and press B can do the same job in 2 hours. If both presses A and B work on the job at the same time, in how many hours can they complete it? _______ | $\frac{6}{5}$ hours

73 Operating alone, it takes press A _______ hours to complete the job. | 3

74 In one hour, press A completes _______ (what fraction) of the job. | $\frac{1}{3}$

75 In one hour, press B completes _______ of the job. | $\frac{1}{2}$

In h hours, A does $\frac{h}{3}$ of the job and B

76 does _______ of the job. | $\frac{h}{2}$

Now, if h is the number of hours it takes A and B together to complete the job, then in h hours one job is completed.

77	That is, $\frac{h}{2} + \frac{h}{3} =$ _______.	1
	We can solve for h:	
	$3h + 2h = 6$	
78	$h =$ _______	$\frac{6}{5}$

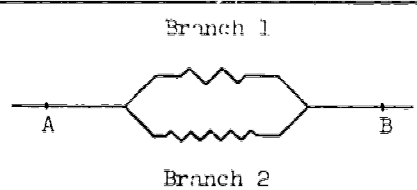
Presser A and C, working together, can complete the same job in 2 hours. How long would it take

79 C alone? _______ 6 hours

You probably noted that A completes $\frac{2}{3}$ of the

80 job in 2 hours. Thus, C completes ______ of $\frac{1}{3}$

81 the job in 2 hours, or _____ of it in 1 hour. $\frac{1}{6}$



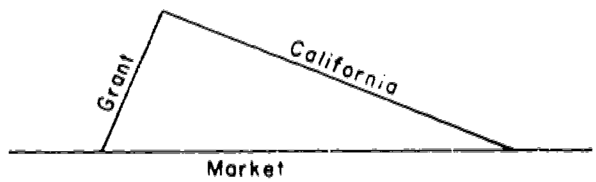
In the portion of the electric circuit shown, the reciprocal of the resistance between points A and B equals the sum of the reciprocals of the resistances of the branches. If the total resistance is 2.4 units, and Branch 1 has a resistance of 3.2 units, what is

82 the resistance of Branch 2 ? _____ units 9.6 units

(See page viii if you need help.)

[illegible]

[illegible] to Market. [illegible] How far is it along Grant between California and Market?



	If we let n be the number of blocks along Grant,	
1	then the number of blocks along Market is ______.	$\sqrt{8^2 + n^2}$
	His total walk is ______ blocks.	$n + 8$
3	The walk along Market is, therefore, $(n + 8) - 4$ or ______ blocks.	$n + 4$
4	We are led to the equation ______.	$\sqrt{8^2 + n^2} = n + 4$

How shall we solve $\sqrt{8^2 + n^2} = n + 4$? Our usual procedures involve adding a real number or multiplying by a non-zero real number to obtain a simple equation. Neither of these techniques will "get rid" of the square root. We are tempted to try "squaring both sides". Is this a reversible process? We cannot be sure, but let us try "squaring" anyway.

	If we "square both sides" of $\sqrt{8^2 + n^2} = n + 4$ we obtain	
5	______ $= (n + 4)^2$	$8^2 + n^2$
6	$64 + n^2 =$ ______.	$n^2 + 8n + 16$
	Now we may solve $64 + n^2 = n^2 + 8n + 16$.	
7	$48 =$ ______.	$48 = 8n$
8	The solution of this final sentence is ______.	6

2..

Since we are not sure that our steps are reversible, we check in the original equation

9 $\sqrt{8^2 + 6^2} = \sqrt{64 + 36} =$ ______, 10

10 and $6 + 4 =$ ______. 10

Therefore we may conclude that the Grant Avenue portion

11 of the walk is ______ blocks. 6

In this case, "squaring both sides" yielded a solution of the original problem.

Although we were successful in our approach to the last example, we need to investigate whether squaring both sides of an equation always leads to an equivalent equation.

Let us start with the simple equation $x = 3$, having

12 the obvious solution set ______. $\{3\}$

If we square both sides of $x = 3$, we have

13 $x^2 =$ ______ $x^2 = 9$

14 The truth set of $x^2 = 9$ is ______. $\{-3,3\}$

15 $x = 3$ ______ (is, is not) equivalent to $x^2 = 9$. **is not**

We are led to the conclusion that squaring both members of an equation may lead to a new equation

16 with a ______ (larger, smaller) truth set. **larger**

Now square both members of $x = 0$.

17 We obtain the equation ______. $x^2 = 0$

18 The truth set of $x^2 = 0$ is the same as the truth set of $x = 0$. $\{0\}$

We see that we shall have to be careful in drawing conclusions about the truth set when we square both sides of an equation. It may be that by squaring we create a new equation which has more elements in its truth set than the original one had.

19	If a and b are two real numbers such that $a^2 = b^2$, can we conclude that $a = b$? [A] yes [B] no	

$(2)^2 = (-2)^2$ is a true sentence. But is the sentence $2 = -2$ true? [B] is correct.

In fact, we can correctly reason as follows: If it is true that

$a^2 = b^2$, then

$a^2 - b^2 = 0$	is true, and
$(a - b)(a + b) = 0$	is true, and
$a - b = 0$ or $a + b = 0$	is true, and
$a = b$ or $a = -b$	is true.

If we begin with the sentence "$a = b$ or $a = -b$", we can reverse these steps and obtain $a^2 = b^2$. Therefore, $a^2 = b^2$ if and only if $a = b$ or $a = -b$.

20	The equation $x - 1 = 1$ has the truth set ______.	{2}
21	$(x - 1)^2 = 1^2$ is equivalent to $x - 1 = 1$ or $x - 1 =$ ______.	$x - 1 = 1$ or $x - 1 = -1$
22	The truth set of $(x - 1)^2 = 1$ is ______.	{0,2}

23	We know that <u>if</u> $a = b$, <u>then</u> $a^2 =$ ______.	$a^2 = b^2$
24	Hence, any solution of a given equation is also a ______ of the equation obtained by squaring both sides of the given equation.	solution
25	Thus, since 4 is a solution of $3x - 5 = 7$, it is also a ______ of $(3x - 5)^2 = (7)^2$.	solution

As a result of our discussion, we conclude that whenever we square both members of an equation in attempting to discover its solution set, we must check each solution of the new equation in order to be sure it is a solution of the original equation.

Checking is absolutely necessary.

Squaring both sides ______ (is, is not) always an equivalent [illegible]	is not
[illegible] not necessarily [illegible] an ______ equation.	equivalent
Hence, each solution of the new equation must be checked to see if it is a ______ of the original equation.	solution

We leave this section with a problem which leads to equations involving a square root. It is in solving such equations that we make the greatest use of squaring both sides. We shall examine some equations of this type.

In first consideration [illegible] remember that the symbol $\sqrt{a}$ names a real number if and only if ______.	$a \geq 0$
The sentence $\sqrt{x-1} = 3$ is equivalent to $\sqrt{x-1} = 3$ and $x \geq$ ______.	1
Squaring both sides of $\sqrt{x-1} = 3$, we have	
______	$x - 1 = 9$
x ______	$x = 10$
33 We must check. Is 10 a solution of $\sqrt{x-1} = 3$? ______ (yes, no)	yes

We must also remember that if a is a non-negative	
34 real number, then $\sqrt{a}$ is also a ______ real number. Further, $\sqrt{a^2} = \lvert a \rvert$.	non-negative
Solve each of the following:	
35 $\sqrt{x} = 3$	{9}
36 $\sqrt{x} = -3$	$\emptyset$
37 $-\sqrt{x} = 3$	$\emptyset$
38 $-\sqrt{x} = -3$	{9}

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Here are some less obvious examples. Solve each equation. If you need help, see pages viii and ix.

39	$\sqrt{x^2 - 16} = 8 - x$	$\{5\}$
40	$\sqrt{x^2 - 16} = x - 8$	$\emptyset$
41	$\sqrt{x^2} = x$	Set of non-negative real numbers
42	$\sqrt{x} = 2 - x$	$\{1\}$
43	$\sqrt{2x} = 1 + x$	$\emptyset$
44	$2\sqrt{x - 2} = x - 1$	$\{3\}$

In order to solve $\sqrt{x} + x = 2$, we wish to obtain an equation without radicals. If we square both sides, we have

$$(\sqrt{x} + x)^2 = 2^2$$

$$x + 2x\sqrt{x} + x^2 = 4$$

This last equation still contains a radical.

To solve $\sqrt{x} + x = 2$ we could first write the equivalent equation

$$\sqrt{x} = 2 - x.$$

45 Solve $\sqrt{x} + x = 2$. $\{1\}$ [See Item 42.]

Solve each of the following. Answers are on page ix.

46. $\sqrt{4x} - x + 3 = 0$
47. $\sqrt{2x + 1} = x + 1$
48. $\sqrt{x + 1} - 1 = x$
49. $2\sqrt{1 + x^2} = 1 + 2x$

Squaring both sides of an equation is also a useful technique to apply to equations involving absolute value. You will recall that for any real number a, $\sqrt{a^2} = |a|$.

$\sqrt{x^2} = |x|$ is true for all real numbers x.

We might square both sides and obtain

50 $\underline{\qquad\qquad} = |x|^2$. $x^2 = |x|^2$

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Is $x^2 = |x|^2$ true for all real numbers x? (A) yes (B) no	

We check that $x^2 = |x|^2$ is true if $x < 0$, if $x = 0$, and finally, if $x > 0$. The correct choice is [A].

We may make use of $x^2 = |x|^2$ in solving	
$|x| = |x + 1|$.	
Squaring both members, we have	
x^2 ________.	$x^2 = x^2 + 2x + 1$
x ________.	$x = -\frac{1}{2}$
[illegible]	is
The solution set of $|x| = |x + 1|$ is ________.	$\{-\frac{1}{2}\}$

Solve each of the following equations. If you need help, see page x.	
[illegible]	[illegible]
$x = |x| - 1$ [illegible]	$|x - 2|$ [illegible]

The distance between x and 3 is 2 more than x. x is ________.	$\frac{1}{2}$

If you were unable to respond correctly [illegible], see page x.

The time t, in seconds, required for a falling body to fall [illegible] feet from a position of rest is given by the formula $t = \sqrt{\frac{2s}{g}}$. If t is [illegible] seconds and g is 32, then s is ________.	625

The hypotenuse of a right triangle is 4 inches less than the sum of the lengths of the legs. If one leg is 12 inches, what is the length of the hypotenuse?

[See answer below.]

If h represents the length of the hypotenuse in inches, then the third side is $\sqrt{h^2 - 144}$ inches long. An open sentence is $h + 4 = \sqrt{h^2 - 144} + 12$. We find $h = 13$. The hypotenuse is 13 inches long.

[illegible] Inequalities

We have had experience in solving inequalities by obtaining simpler equivalent inequalities. Recall that in working with inequalities we make use of [illegible] properties of order.

For real numbers a, b, c,

if $a < b$ then $a + c < b + c$,

and

for c positive, if $a < b$ then $ac < bc$,

for c negative, if $a < b$ then $ac > bc$.

	[illegible] of the following, find a sentence of the form $x < a$ or of the form $x > a$ which is equivalent to the [illegible] inequality. If you need help, see page xi.		
1	x [illegible]	$x <$ ____	$x < 6$
2	$\frac{x}{4}$ [illegible]	x ____	$x > 28$
3	$-3x$ [illegible]	____	$x < -18$
4	[illegible]x - [illegible] $> x + 4$	____	$x > 3$

Solve

$$\frac{4}{5}y - 6 < \frac{2}{3}y + \frac{5}{6}$$

One procedure is to start by "getting rid of the
5 fractions" by multiplying by the positive number ______. 30

6 This step ______ (is, is not) reversible and leaves the order of the resulting products unchanged. is

Multiplying, we have

7 $$24y - 180 \text{ ______ } 20y + 25.$$ $<$

Next we may rewrite this last sentence as:

8 $24y - 20y < 25$ ______. The step is reversible. $25 + 180$

$4y < 205.$

Now we may divide by the positive number 4 and obtain

9 y ______. $y < \frac{205}{4}$

10 Notice that each step is ______. reversible

11 Write out all the steps in solving $4 - \frac{x}{2} > x - \frac{1}{2}$. [See page xi.]

12 Write out all the steps necessary to reverse your work of Item 11. [See page xi.]

Although it is usually convenient to "get rid of the fractions" as a first step in solving an inequality such as

$$\frac{t}{4} + \frac{2}{3} < \frac{4}{3} + \frac{t}{6},$$

this is not the only way to begin. For instance, we might proceed as follows:

$$\frac{t}{4} + \frac{2}{3} < \frac{4}{3} + \frac{t}{6}$$

$$\frac{t}{4} - \frac{t}{6} < \frac{4}{3} - \frac{2}{3}$$

$$\frac{t}{12} < \frac{2}{3}$$

$$t < 8$$

Notice that all the steps are reversible.

A [illegible] question is: What about checking? If we start with one inequality [illegible] a simpler one, we may "check" by [illegible] each step. There is a danger, however, that we may make the same [illegible] errors in [illegible] directions.

	[illegible] "spot check".	
	We assert that $t <$ [illegible] is equivalent to	
	$\frac{t}{[illegible]} - \frac{[illegible]}{[illegible]} < \frac{[illegible]}{[illegible]} + \frac{t}{[illegible]}$.	
[illegible]	[illegible] ______ (is, is not) in the solution set of $t <$ [illegible].	is
	If t [illegible], then $\frac{t}{[illegible]} + \frac{[illegible]}{[illegible]}$ is ______ and	$\frac{2}{3}$
	$\frac{[illegible]}{3} + \frac{[illegible]}{[illegible]}$ is ______.	$\frac{4}{3}$
	$\frac{[illegible]}{[illegible]} < \frac{[illegible]}{[illegible]}$ is ______ (true, false).	true
[illegible]	Therefore, [illegible] ______ (is, is not) in the solution set of	is
	$\frac{[illegible]}{[illegible]} + \frac{[illegible]}{3} < \frac{[illegible]}{[illegible]} + \frac{[illegible]}{[illegible]}$.	
	[illegible] is not a solution of $t <$ [illegible].	
18	Is [illegible] a solution of $\frac{t}{[illegible]} + \frac{[illegible]}{3} < \frac{[illegible]}{3} + \frac{[illegible]}{[illegible]}$? ______	no

This sort of "checking" is not a complete verification of our result, but it is often a useful device. In particular, is [illegible] a solution?

Solve the following inequalities. Write your steps neatly to avoid mistakes.

	Inequality	Truth Set	
19	[illegible]$x <$ [illegible] $- x$	The set of ______	real numbers less than 21
20	$\sqrt{2}$ [illegible] $2x > 3\sqrt{2}$	The set of ______	real numbers greater than $\sqrt{2}$
21	$t\sqrt{3} < 3$	The set of ______	real numbers less than $\sqrt{3}$
22	[illegible]$y - 3 > 3y +$ [illegible]	The set of ______	real numbers greater than 2

To solve $1 < 2x + 1 < 3$, we first recall that this sentence is equivalent to the compound sentence

$$1 < 2x + 1 \quad \underline{\text{and}} \quad 2x + 1 < 3.$$

We may solve this compound sentence as follows:

$$\begin{array}{lcl} 1 < 2x + 1 & \text{and} & 2x + 1 < 3 \\ 0 < 2x & \text{and} & 2x < 2 \\ 0 < x & \text{and} & x < 1 \end{array}$$

The solution set of $1 < 2x + 1 < 3$ is the set of all real numbers between 0 and 1. This solution set is:

Graph the truth set of the following inequalities. Answers are on pages xi and xii.

23. $1 < 4x + 1 < 2$

24. $4t - 4 \le 0$ and $1 - 3t < 0$

25. $-1 \le 2t < 1$

26. $6y + 3 < 0$ or $6y - 3 > 0$

27	$\|x\|$ is the distance between x and ______.	0
28	Hence, if x is between $-\frac{1}{2}$ and $\frac{1}{2}$, $\|x\|$ ____ $\frac{1}{2}$.	$\|x\| < \frac{1}{2}$
	Similarly, $\|x - 1\|$ is the distance between x	
29	and ______.	1
	Thus, we may interpret $\|x - 1\| < 2$ as	
30	"The ____ between x and 1 is less than ____."	distance, 2
31	Graph the truth set of $\|x - 1\| < 2$. ____	

32 Which of the following sentences is equivalent to $|x - 1| < 2$?

[A] $-1 < x < 3$ [B] $x - 1 < 2$

$|x - 1| < 2$ and $-1 < x < 3$ have the same truth set; namely, the set of all numbers between -1 and 3. Hence, [A] is correct. Notice that $x - 1 < 2$ and $|x - 1| < 2$ have <u>different</u> truth sets.

33 | $|x - a|$ is the distance between x and ______. | a

Since $|x + [illegible]| = |x - (-[illegible])|$, $|x + [illegible]|$ is the distance

34 | between x and ______. | -2

The solution set of $|x + [illegible]| < [illegible]$ is the set of all

35 | real numbers between ______ and ______. | [illegible] and [illegible]

The next few responses [illegible] with inequalities of the form [illegible], where [illegible].

36 | We have found that they are ______ to compound | equivalent

37 | sentences of the form "[illegible] and ______." | [illegible]

(We have sometimes written [illegible] as [illegible].)

There is another type of compound sentence with which we are familiar.

This [illegible].

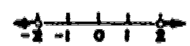
Does [illegible] belong to the truth set of "[illegible] or

38 | x [illegible]"? ______ (yes, no) | yes

The solution set of "[illegible] or [illegible]" is the set of numbers which are either less than

39 | ______ or greater than ______. | $-\frac{1}{2}$, $\frac{1}{2}$

40 | $|x| > [illegible]$ means that either $x <$ ______ or $x >$ ______. | $x < -2$ or $x > 2$

41 | Graph the truth set of $|x| > [illegible]$. ______ | 

$|y + [illegible]|$ can be interpreted as the distance between

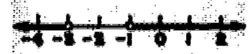
42 | y and ______. | -2

43 | If $|y + 2| > 1$, then the ______ between y and -2 is greater than 1. | distance

44 | -2 ______ (is, is not) an element of the truth set of $|y + 2| > 1$. | is not

The two points which are one unit from -2 are

45 | ______ and ______. | -3 and -1

46 The truth set of $\|y + 2\| > 1$ is the set of all real numbers which are either ____ or ____. Graph the solution set. ______	less than -3 or greater than -1 
If we know that the area of a rectangle is 12 square inches and the length is less than 5 inches, then what can we say about its width? If the rectangle is w inches wide, then since the area is ______ square inches, the length is ______. Then, $\frac{12}{w} <$ ____ and w ____ $\frac{12}{5}$. Note that we must have w ____ 0 (>,<) to satisfy the conditions of the problem.	12; $\frac{12}{w}$; $\frac{12}{w} < 5$, $w > \frac{12}{5}$; $w > 0$
[illegible], then $cx < d$ is equivalent to x ____ $\frac{d}{c}$. (<,>)	$x < \frac{d}{c}$
[illegible], then $cx < d$ is equivalent to x ____ $\frac{d}{c}$. (<,>)	$x > \frac{d}{c}$

The difficulty, as you expect, arises if we wish to multiply or divide both sides of an inequality by a phrase containing a variable. Some phrases, [illegible], are positive for all values of the variable. Others, such as [illegible], are negative for all values of the variable.

Which of the following is a negative real number for every value of the variable x?
[A] $-x$ [C] $\frac{1}{-x^2 - 1}$
[B] $\|-x - 1\|$ [D] $-\|x + 1\|$

[C] is correct. You should have noted that $-x$ is positive if x is negative. Also, $|-x - 1|$ cannot be negative, whatever the value of x. Finally, $-|x + 1|$ is negative except when x is -1.

55	Which one of the following phrases is positive for every value of the variable? [A] $\lvert -(x^2 + 1) \rvert$ [B] $(-x - 4)^2$ [C] $x + 1$	

$(-x - 4)^2$ is 0 if $x = -4$. $x + 1$ is negative if $x < -1$. Notice that since $-(x^2 + 1)$ is non-zero for all values of x, it follows that $\lvert -(x^2 + 1) \rvert > 0$. [A] is the correct choice.

56	To solve $\frac{1}{x^2 + 1} < 1$, we observe that $x^2 + 1$ is ____ for all values of the variable.	positive
57	Hence, $\frac{1}{x^2 + 1} < 1$ is equivalent to ____ $< x^2 + 1$.	$1 < x^2 + 1$
58	This last inequality leads to $0 <$ ____.	$0 < x^2$
59	The truth set is the set of all real numbers except ___.	0
	Solve $-\frac{2}{x^2 + 2} \geq -1$.	
60	The truth set is the set of ______.	all real numbers [see page xii.]

Unfortunately, many phrases involving variables are positive for some values of the variable, zero for some values, and negative for still other values. Inequalities which are solved by multiplying by such phrases offer a new challenge. The remainder of this section is starred. It deals with some problems of this latter type and with related ideas.

* How do we solve an inequality such as

$$(x - 1)(x - 3) > 0 \text{ ?}$$

Notice first that 1 and 3 are not solutions. Let us select one factor, say, $x - 1$, and argue as follows:

If $x - 1 > 0$, then we divide by $x - 1$ and obtain $x - 3 > 0$. In other words, if $x - 1 > 0$, we are led to the compound sentence

$$x - 1 > 0 \ \underline{\text{and}} \ x - 3 > 0.$$

If $x - 1 < 0$, then we divide by $x - 1$ and obtain $x - 3 < 0$. In other words, if $x - 1 < 0$, we are led to the compound sentence

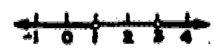
$$x - 1 < 0 \ \underline{\text{and}} \ x - 3 < 0.$$

The truth set of "$x - 1 > 0$ and $x - 3 > 0$" is the
*61 set of real numbers greater than _______. 3

The truth set of "$x - 1 < 0$ and $x - 3 < 0$" is the
*62 set of real numbers less than _______. 1

The truth set of $(x - 1)(x - 3) < 0$ is the union of the truth sets of the compound sentences in Items *61 and *62.

Therefore, the truth set of $(x - 1)(x - 3) < 0$ is the
*63 union of the set of real numbers less than _______ and 1
*64 the set of real numbers greater than _______. 3

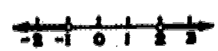
*65 Graph $(x - 1)(x - 3) > 0$. _______ 

The argument which we have used may be interpreted in the following way:

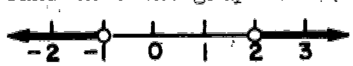
if a and b are real numbers, and
if $ab > 0$, then either $a > 0$ and $b > 0$
or $a < 0$ and $b < 0$.

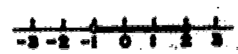
*66 Graph $(x - 2)(x - 5) > 0$ _______ 

*67 Graph $x(x - 4) > 0$ _______ 

*68 Graph $(x + 1)(x - 2) > 0$ _______ 

We have found that the graph of $(x + 1)(x - 2) > 0$ is



*69 The graph of $(x + 1)(x - 2) < 0$ is _______. 

*70 The graph of $(x + 1)(x - 2) \leq 0$ is _______. 

* We might approach the problem of solving $(x + 1)(x - 2) < 0$ by using the following:

> If a and b are real numbers, and
> if $ab < 0$, then either $a > 0$ and $b < 0$
> or $a < 0$ and $b > 0$.

If $(x + 1)(x - 2) < 0$ is true for some x,

*71 then either $x + 1 > 0$ and ______ | $x - 2 < 0$

*72 or ______ and ______ for the same x. | $x + 1 < 0$ and $x - 2 > 0$

We might proceed by first noticing that if x is -1

*73 or 2, then $(x + 1)(x - 2)$ ______ 0. We have indicated this on the number line below by writing "o" above the points whose coordinates are -1 and 2. | $=$

```
            o         o
 -4  -3  -2  -1   0   1   2   3   4
```

We may consider the line as separated into three

*74 regions: points to the left of ______, | -1

*75 points between -1 and ______, | 2

*76 and points to the ______ of 2. | right

For points to the left of -1, $x + 1 < 0$ and $x - 2 < 0$.

*77 Therefore, $(x + 1)(x - 2)$ ______ 0. | $>$

*78 For points between -1 and 2, $x + 1 > 0$ and ______. | $x - 2 < 0$

*79 Therefore, $(x + 1)(x - 2)$ ______ 0. | $<$
(See Item *71.)

For points to the right of 2, $x + 1 > 0$ and $x - 2 > 0$.

*80 Therefore, $(x + 1)(x - 2)$ ______ 0. | $>$

Notice that there are no points for which "$x + 1 < 0$ and $x - 2 > 0$" is a true sentence.

---

* We might draw the following diagram showing regions where $(x + 1)(x - 2)$ is negative, zero and positive.

```
 <--   + + + 0 - - - - - 0 + + + -->
 -4  -3  -2  -1   0   1   2   3   4
```

With such a diagram we may read off the truth sets of the four inequalities:

$(x+1)(x-2) < 0$, $(x+1)(x-2) \le 0$, $(x+1)(x-2) > 0$, $(x+1)(x-2) \ge 0$.

Construct a similar diagram for each of the following. For answers, see page xii.

*81. $(x - 1)(x - 4)$ ______ *83. $(x + 3)(x + 1)$ ______

*82. $x(x - 2)$ ______ *84. $x^2(x - 3)$ (Careful!) ______

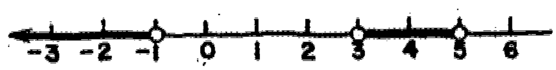
Using your responses, complete a simpler open sentence for each of the following:

	Inequality	Equivalent Sentence	
*85	$(x - 1)(x - 4) > 0$	$x < 1$ or x ____	$x < 1$ or $x > 4$
*86	$x(x - 2) \leq 0$	$0 \leq x \leq$ ____	$0 \leq x \leq 2$
*87	$(x + 3)(x + 1) = 0$	$x = -3$ or ____	$x = -3$ or $x = -1$
*88	$x^2(x - 3) \geq 0$	$x =$ ____ and $x \geq$ ____	$x = 0$ and $x \geq 3$

*89 Try to follow a similar procedure and graph the truth set of

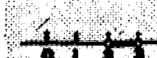
$$(x + 1)(x - 3)(x - 5) < 0.$$

Answer below

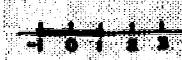


Graph the inequalities given below.

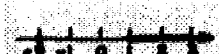
*90 $y^2 - 5y + 6 < 0$ ______



*91 $y^2 \leq 1$ ______



*92 $(x + 2)^2(x - 1) \geq 0$ ______



19-5. Summary and Review

In this course we have often worked with equivalent open sentences. Two sentences are equivalent if they have the same truth set.

In finding the truth set of an open sentence we often look for an equivalent sentence whose truth set is obvious.

In the case of equations, two operations which yield equivalent equations are:

(1) adding a real number to both members,

(2) multiplying both members by a non-zero real number.

Some operations which yield equivalent inequalities are:

(1) adding a real number to both members,

(2) multiplying both members by a positive number, in which case the order of the resulting products is unchanged,

(3) multiplying both members by a negative number, in which case the order of the resulting products is reversed.

If the left member of an equation is a product of polynomials and the right member is 0, then we can often apply the property of real numbers that:

For real numbers a and b, $ab = 0$ if and only if $a = 0$ or $b = 0$.

We can apply our general knowledge of equations in solving equations involving fractions. It is very important that we note carefully the domain.

Squaring both members of an equation is sometimes useful. However, this operation may not result in an equivalent equation. Consequently, we must check each solution of the new equation in order to identify the solutions of the original equation.

Review Problems

In problems 1-20, find the truth set of each equation.

1. $x(x^2 + 1) - 3(x^2 + 1) = 0$
2. $(x - 3) \cdot \frac{x^2 - 1}{x^2 - 1} = 2$
3. $(x - 3)(x^2 + 4) = 2(x^2 + 4)$
4. $\frac{x}{x - 3} - \frac{3}{x - 3} = 0$
5. $\frac{x + 2}{x - 2} = 0$
6. $\frac{|x|}{2} = 1$
7. $-\frac{2x}{3} + \frac{1}{5} = x - \frac{1}{15}$
8. $\frac{x}{x - 3} + \frac{3}{x - 3} = 0$

9. $(x + 1)(x - 3) = 7(x - 3)$

10. $(x + 1)(x^2 - 2) = -(x + 1)$

11. $x(x - 1)(x - 2) = 0$

12. $\dfrac{2}{x - 8} = \dfrac{1}{x - 2}$

13. $\dfrac{1}{x} - 3 + \dfrac{2x - 1}{x} = 0$

14. $\sqrt{x + 2} + 2 = 0$

15. $\sqrt{x + 2} - 2 = 0$

16. $|x + 1| = 3$

17. $|x| + x = 1$

18. $|x| + 1 = x$

19. $\dfrac{x + 1}{x + 1} = 1$

20. $\dfrac{x^2 + 1}{x^2 + 1} = 1$

21. Solve and graph the following sentences.

(a) $\dfrac{x + 2}{x - 2} = 0$

*(b) $\dfrac{x + 2}{x - 2} > 0$

*(c) $x^2 - 4 > 0$

(d) $|x| > 2$

*22. Graph the truth set of each of the following sentences.

(a) $(x - 3)(x - 1)(x + 1) > 0$

(b) $(x - 3)(x - 1)(x + 1) > 0$ and $x \geq 0$

(c) $(x - 3)(x - 1)(x + 1) > 0$ or $x \geq 0$

23. A man makes a trip of 300 miles at an average speed of 30 miles per hour and returns at an average speed of 20 miles per hour. What was his average speed for the entire trip?

*24. Generalizing Problem 23. A man makes a trip of d miles at an average speed of r miles per hour and returns at an average rate of q miles per hour; what was his average rate for the entire trip?

25. One automobile travels a distance of 360 miles in 1 hour less than a second going 4 miles per hour slower than the first. Find the rate of the two automobiles.

26. One leg of a right triangle is 2 feet more than twice the shorter leg. The hypotenuse is 13 feet. What are the lengths of the legs?

*27. Find the truth set of $|x - 5|^2 \geq 9$.

*28. At what time between 3 and 4 o'clock will the hands of a clock be together?

Chapter 20

THE GRAPH OF $Ax + By + C = 0$

20-1. The Real Number Plane

Consider the following open sentence:

$$3y - 2x + 6 = 0.$$

What would we mean by the truth set of this sentence?

Let us consider first, the open sentence in one variable,

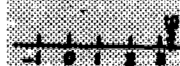
$$3y - 10 = 0.$$

1 The truth set for this sentence is ______. $\{\frac{10}{3}\}$

Therefore, if y has the value $\frac{10}{3}$, then

2 $3y - 10 = 0$ is ______ (true, false). true

3 The graph of $3y - 10 = 0$ is ______.



Remember that the graph of the sentence $3y - 10 = 0$ is the graph of the truth set of this sentence.

We are able to graph this sentence since every point on the number line corresponds to a real number.

4 Now let us go back to the problem of finding the truth set of

$$3y - 2x + 6 = 0.$$

Clearly, the truth set must contain values of the variables x and y which make this sentence true. Suppose we try to assign the values 0 and -2 to the variables x and y. Which of the following sentences would we have?

P. $3(0) - 2(-2) + 6 = 0$

Q. $3(-2) - 2(0) + 6 = 0$

[A] Both P and Q.

[B] Either P or Q.

[C] I can't answer this!

Since nothing has been said about which value to assign to x and which value to assign to y, we don't know whether P or Q is the intended sentence. We need to know exactly what we mean before we can test for members of the truth set. The correct answer is [C].

In the open sentence

$$3y - 2x + 6 = 0$$

assign the value -2 to the variable x and the value 0 to the variable y. The open sentence

5 becomes $3(\quad) - 2(\quad) + 6 = 0$. — $3(0)-2(-2)+6=0$

6 This is a ______ (true,false) sentence. — false

If we assign the value 0 to the variable x, and the value -2 to the variable y, the open sentence becomes

$$3(-2) - 2(0) + 6 = 0.$$

7 This is a ______ (true,false) sentence. — true

The pair of values, 0 for x and -2 for y,

8 makes the sentence ______. — true

The pair of values, -2 for x and 0 for y,

9 makes the sentence ______. — false

Now let us try another pair of values for the variables x and y. Let x have the value 2 and y have the value $-\frac{4}{3}$.

$$3(-\frac{4}{3}) - 2(1) + 6 = 0$$

10 is a ______ (true,false) sentence. — true

It should be evident from the above discussion that the truth set of an open sentence in two variables will contain pairs of numbers. Each pair of numbers will consist of a value for the variable x and a value for the variable y. The truth set will be the set of all pairs which make the sentence true.

It is awkward to keep writing "______ is the value of x and ______ is the value of y." We would like to use a notation that would indicate

1) pairs of numbers,
2) which number is the x-value and which number is the y-value.

We agree to write $(0,-2)$ to mean x has the value 0, and y has the value -2. The <u>order</u> in which we write the numbers 0 and -2 in ten notation $(0,-2)$ is important. Thus, we are considering <u>ordered pairs of real numbers</u>. Note that we write an ordered pair enclosed in parentheses with the numbers separated by a comma.

	The ordered pair $(0,-2)$ is in the truth set of the equation $$3y - 2x + 6 = 0$$	
11	since $3(-2) - 2(0) + 6 = 0$ is a ______ (true, false) sentence.	true
12	The ordered pair $(1,-\frac{4}{3})$ ______ (is, is not) in the truth set of the equation $$3y - 2x + 6 = 0.$$	is
13	The ordered pair $(-\frac{4}{3},1)$ ______ (is, is not) in the truth set of the equation $$3y - 2x + 6 = 0$$	is not
14	since $3(\ \ ) - 2(\ \ ) + 6 = 0$ is a	$3(1)-2(-\frac{4}{3})+6=0$
15	______ (true, false) sentence.	false

	Since $3(-2) - 2(0) + 6 = 0$ is a true sentence the	
16	ordered pair $(\ \ ,\ \ )$ is in the truth set of this equation.	$(0,-2)$
17	The ordered pair $(-2,0)$ ______ (is, is not) in the truth set of the open sentence.	is not

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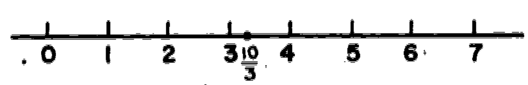
	The truth set of an open sentence in two variables	
18	consists of ______ pairs of real numbers which	ordered pairs
	make the sentence true.	
	We always write the ordered pair as	
19	(value of first variable, ______ of second variable).	value
20	An ordered pair of real numbers is in the ______	truth, or solution
	set if it satisfies the sentence.	

For any equation in the variables x and y, we agree always to call x the first variable, and y the second variable.

21	If we write $(\frac{9}{2},1)$ we mean: x has the value ______	$\frac{9}{2}$
22	and y has the value ______.	1
	If the equation is in two variables other than x	
	and y, we must always specify which variable is to	
23	be considered the ______ variable and which is to	first
	be considered the second variable.	

	If we consider the open sentence	
	$s = r + 1,$	
	and we take r to be the first variable, by the	
24	ordered pair (0, 1) we mean ______ has the value 0.	r
25	______ has the value 1.	s
26	(0,1) ______ (is, is not) in the solution set of the equation	is
	$s = r + 1.$	

We are able to graph, on the number line, the truth set of an equation in one variable. For example, the graph of $3y - 10 = 0$ is



How would we graph the truth set of an equation in two variables? Since the truth set consists of ordered pairs of real numbers, we would need pairs of number lines to represent these solutions.

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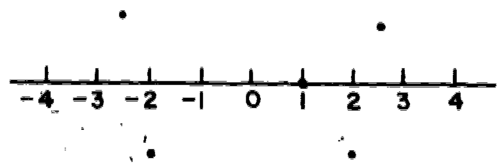
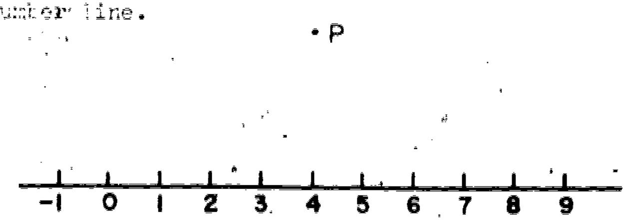
Before we continue with the discussion of the solutions of equations in two variables, we shall discuss the real number plane.

Can we find a way to associate ordered pairs of real numbers with points in the plane?

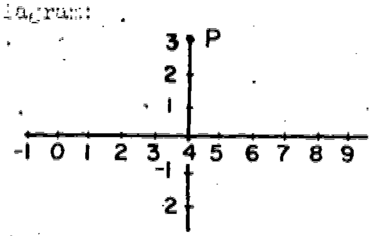


27 In a plane with one real number line drawn, as above, a point will be either on the number line or above the number line or ______ the number line. below

In this figure, P is directly above the point 4 on the number line.



Let us draw a vertical number line through P as shown in the diagram:



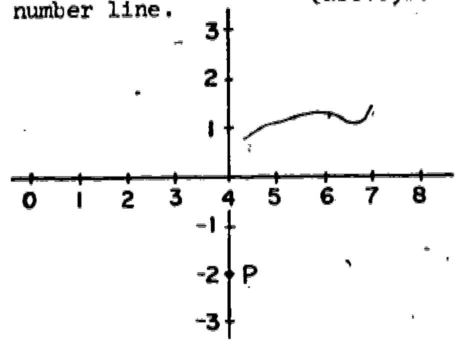
28 We may associate P with 4 on the horizontal ______ line. number

P is also associated with a number on the second or vertical number line; namely, with ______.

29 3

Thus, we associate with the point P the real numbers 4 and 3. The order of writing the pair of numbers is important. By (4, 3) we shall mean a point 4 units to the right of 0 on the horizontal number line and 3 units above 0 on the vertical number line. P is associated with the ordered pair (4, 3).

30 In the diagram shown here P is ________ (above, below) the horizontal number line. — below



In this diagram, P is associated with the number 4

31 on the ______ number line. — horizontal

Hence, the first number of the ordered pair of

32 numbers for P will be ______. — 4

33 P is at the number -2 on the ______ number line. — vertical

34 The ______ number of the ordered pair of numbers — second

35 for P will, therefore, be ______. — -2

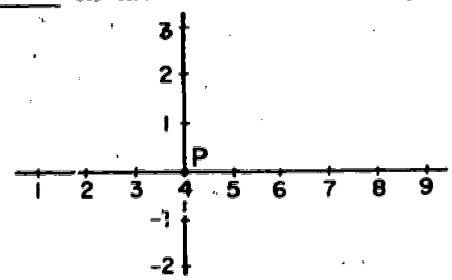
We can write, in this case, as a label for P the

36 ordered pair of numbers (,). — (4,-2)

How would we label a point on the horizontal number line, using an ordered pair of real numbers?

In the figure below, P is associated with the the

37 number ______ on the horizontal number line. — 4



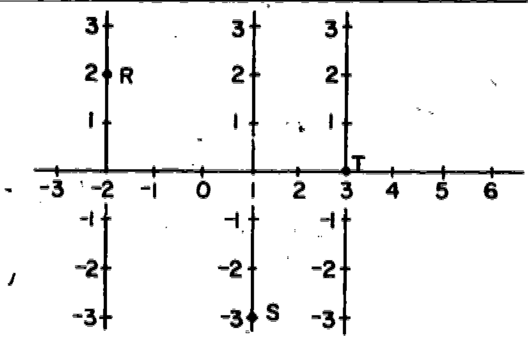
With what number on the vertical number line should we

38 associate P? P is ________ (above, below, on) the horizontal — on

number line. Since P is on the horizontal line, it

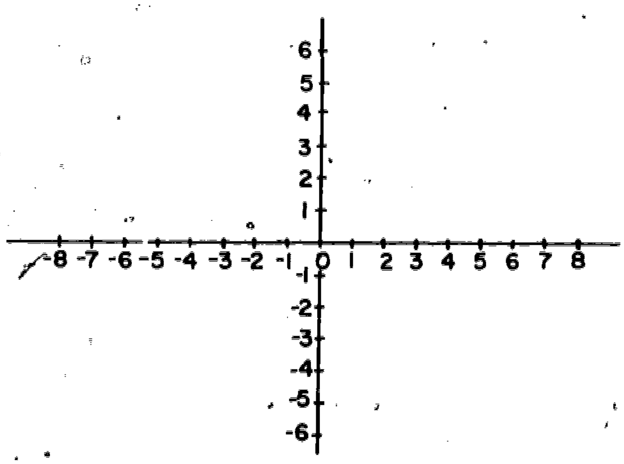
39 would be associated with the number ______ on the vertical number line. — 0

40 Thus, P is associated with the ordered pair (,). — (4,0)



	Through each point R, S and T a vertical	
41	______ ______ is drawn.	number line
	On the horizontal number line, point R is associated	
42	with the number ______.	-2
43	The point R is associated with the number ______ on the vertical number line.	2
44	R is associated with the ordered pair ______.	(-2,2)
	The point S is associated with the ordered pair	
45	______.	(1,-3)
46	The point T is associated with the ______ pair (3,0).	ordered pair

Do we need a separate vertical line through each point in the plane? The preceding items show that each point in the plane is associated with an ordered pair of real numbers. We shall agree to draw only one second number line, perpendicular to the first number line and with the same zero point:



The horizontal and vertical number lines are called coordinate axes.

We usually label the first number line with an "x" and call it the "x-axis". Similarly, it is usual to label the second number line with a "y" and call it the y-axis.

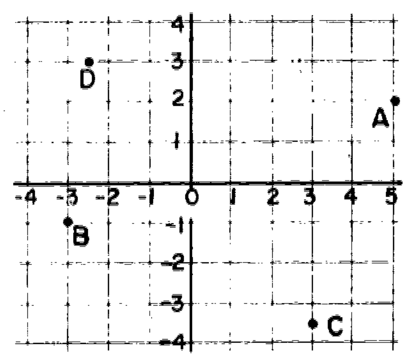
The ordered number pair associated with a point is also called the coordinates of the point.

The first coordinate indicates with what number on the horizontal axis the point is associated; it is called the abscissa of the point.

The second coordinate indicates with what number on the vertical axis the point is associated; it is called the ordinate of the point.

Item		Answer
	The coordinate axes intersect at a point which is	
47	associated with the number ______ on the horizontal	0
48	axis and with the number 0 on the ______ axis.	vertical axis
	This point is called the origin.	
49	Its coordinates are (___ ,0).	(0,0)

Refer to this diagram for Items 50- 62.



Item		Answer
50	The coordinates of point A are ______.	(5,2)
51	5 is the ______ of A.	abscissa
52	The ordinate of A is ______.	2
53	Point B has ______ (-3,-1).	coordinates
	The coordinates of points A, B, C, D are	
54	A(,)	A(5,2)
55	B(,)	B(-3,-1)
56	C(,)	$C(3,-3\frac{1}{2})$
57	D(,)	$D(-2\frac{1}{2},3)$
	Both coordinates of A are positive numbers.	
58	Both coordinates of B are ______ numbers.	negative

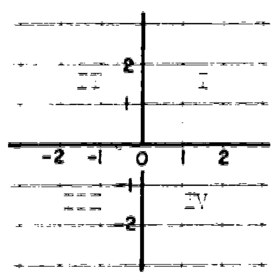
59	The abscissas of _____ and _____ are positive.	A, C
60	The abscissas of B and D are _______.	negative
61	The ordinates of _____ and _____ are negative.	B, C
62	The ordinates of A and D are _______.	positive

We have introduced the terms coordinate axes and coordinates. The coordinate axes divide the plane into four parts called quadrants.

The quadrants are numbered counterclockwise, beginning with the upper right-hand corner where both coordinates are positive. By convention we use Roman numerals for this purpose.

State in which quadrant each of these points lies:

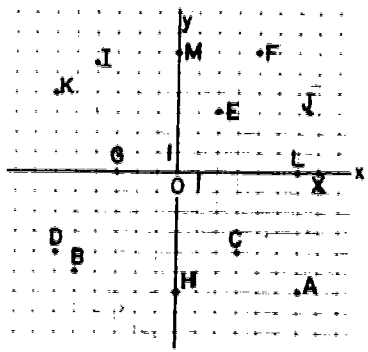
63	(2,-1) _______	IV
64	(-1,2) _______	II
65	(1,2) _______	I
66	(-1,-2) _______	III



In which quadrant(s) are the points with

67	coordinates both positive? _______	I
68	both negative? _______	III
69	abscissas negative and ordinates positive? _______	II
70	one coordinate positive, and the other negative? _______ and _______	II and IV
71	Points with coordinates like (2,0), (0,-3), (-5,0) or (0,0) do not lie in any quadrant. They lie on one or both coordinate _______.	axes
72	If the abscissa is equal to the ordinate, the point is in quadrant _______ or _______, unless it is the origin.	I or III
73	If the abscissa is the opposite of the ordinate, the point is in quadrant _______ or _______, unless it is (0,0).	II or IV

Write the ordered pairs of numbers which are associated with the points A through M in the figure below:



74	A(,)	A(6,-6)
75	B(,)	B(-5,-5)
76	C(,)	C(3,-4)
77	D(,)	D(-6,-4)
78	E(,)	E(2,3)
79	F ____	F(4,6)
80	G ____	G(-3,0)
81	H ____	H(0,-6)
82	I ____	I$(-4,5\frac{1}{2})$
83	J ____	J$(6\frac{1}{2},3)$
84	K ____	K(-6,4)
85	L ____	L(6,0)
86	M ____	M(0,6)

We have associated with any point in the plane an ordered pair of real numbers, called the <u>coordinates</u> of the point.

Recall that we found, in the beginning of this section, ordered pairs of real numbers which satisfied a given equation in two variables. Can we find points in the plane which are associated with these ordered pairs?

Consider the ordered pair (0,-2) in the solution set of the equation

$$3y - 2x + 6 = 0.$$

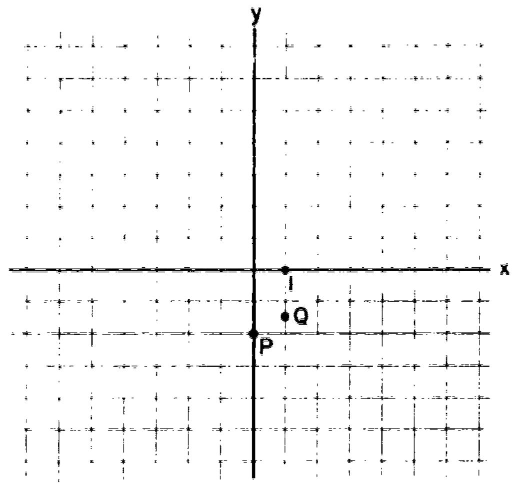
In the plane with x-axis and y-axis as indicated in the following figure, can we locate a point with coordinates (0,-2)?

87 Since 0 is the value for ____ — x

88 and -2 is the value for ____, we locate — y

89 the point P, ____ units from the y-axis — 0

90 and ____ units below the x-axis. — 2



91 The coordinates of P are (,).

92 A point Q, with coordinate $(1,-\frac{4}{3})$ would be ____

93 unit to the ______ of 0 on the x-axis and

94 $\frac{4}{3}$ units $\underset{\text{(above,below)}}{\underline{\qquad\qquad}}$ 0. on the vertical axis;

that is, below the x-axis.

95 The point Q has coordinates (,). $(1,-\frac{4}{3})$

Given any ordered pair of real numbers we can <u>always</u> locate a point in the number plane with this ordered pair as coordinates.

We can associate with any ordered pair of real numbers, exactly one point in the real number plane and with any point in the plane we can associate exactly one ordered pair of real numbers.

96. On your response sheet locate each of the following points and label it with the appropriate capital letter.

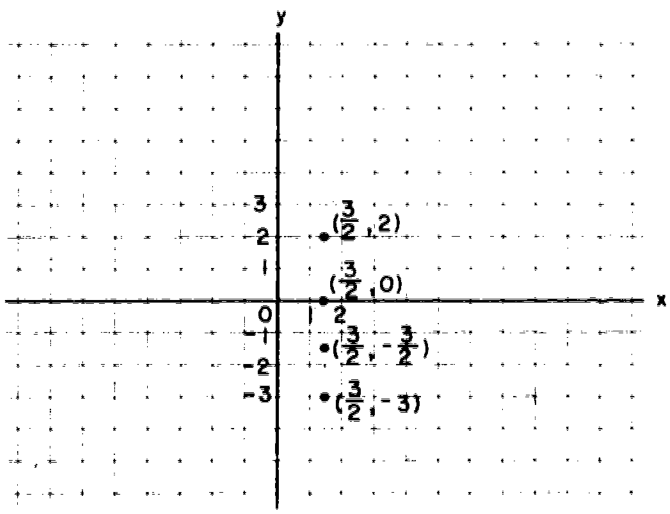
A(1,-3); B(-6,4); C$(0,\frac{8}{3})$; D(-7,-1); E(-4,0);

F(0,0); G$(5,\frac{3}{2})$; H$(\frac{2}{3},5)$; I(-4,-6); J(-6,-4);

K$(0,-\frac{5}{3})$; L$(-\frac{5}{3},0)$.

Turn to page xvi to check your work.

318



Use the above figure to answer Items 97-101.

97 The ________ (abscissa, ordinate) of each point is $\frac{3}{2}$. **abscissa**

98 We could find ________ (a few, many) ordered pairs with abscissa $\frac{3}{2}$. **many**

All points with coordinates in which the abscissas

99 are $\frac{3}{2}$ can be connected by a ______. **line**

100 This line is parallel to the ______ axis and is **vertical axis, or y-axis**

101 _____ units to the right of it. **$\frac{3}{2}$**

Examine the following coordinates:

(10,-3) (2,-3) (0,-3) (-2,-3) (-5,-3)

If these points were located with reference to a set of coordinate axes, they all would be found

102 ________ (above, below) the ______ axis. **below, horizontal axis, or x-axis**

If a line were drawn through these points it would be

103 parallel to the ______ axis and **horizontal axis, or x-axis**

104 ________ (how many) units ________ (above, below) it. **3, below**

In the space provided on the response sheet for Items 105 and 106, draw coordinate axes and mark the following points:

105 A(2, [illegible]); B(2, 1); C(2, [illegible]);

106 D(2,0); E(2, $-\frac{7}{2}$); F(2, -6).

Check with page xvi .

All of the points for which the abscissa is 2 lie

107 on the line parallel to the ______ axis and

108 ______ (how many) units to the ______ of it.

vertical axis, or y-axis

2, right

109 On the response sheet locate several points whose numbered pairs have 5 for their ordinates.

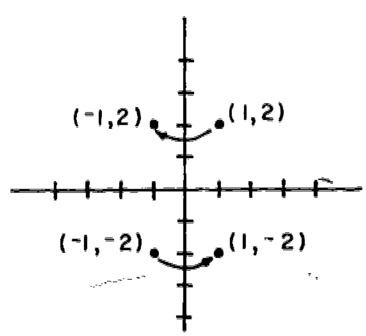
All these points lie on a line,

[A] parallel to the vertical axis and 5 units to the right of it.

[B] parallel to the horizontal axis and 5 units above it.

The correct choice is [B]. Look at page xvii to check your work.

*110. Let us think of moving all the points of a plane in the following manner: Each point with coordinates (c,d) is moved to the point with coordinates (-c,d). Another way of looking at this is to consider that the points of the plane are rotated one-half revolution about the y-axis, as indicated in the figure below.



Answer the following question and locate the points referred to in parts (a) and (b). See page for the answers.

(a) To what points do the following points go:
$(2,1)$, $(2,-1)$, $(-\frac{1}{2},2)$, $(-1,-1)$, $(3,0)$, $(-6,0)$, $(0,4)$, $(0,-4)$?

(b) What points go to the points listed in (a) above?

(c) What point does $(c,-d)$ go to?

(d) What point does $(-c,d)$ go to?

(e) What point goes to (c,d)?

(f) What points go to themselves?

*111 Suppose we change the rule for moving the points to the following: The point (c,d) is moved two units to the right. Then the point goes to $(c+2, d)$. What point does $(-a,b)$ go to?

[A] $(-a+2, b)$
[B] $(2-a, b)$
[C] $(-a-2, b)$
[D] $(a+2, b)$

The correct choice is [A] or [B]. If we add 2 to the abscissa of $(-a, b)$, we find $(-a, b)$ goes to $(-a+2, b)$. This is equivalent to $(2-a, b)$.

20-2. The y-Form of the Equation of a Line

In the preceding section we saw that the truth set of an equation in two variables consists of ______ pairs of real numbers. — **ordered pairs**

For any ordered pair of real numbers, the ordered
2 pair may or may not belong to the ______ set of a — **truth set**
3 given equation in two ______. — **variables**

Thus an open sentence in two variables sorts the set of all ordered pairs of real numbers into two subsets:
4 the set of all ordered pairs that make the ______ — **sentence**
5 true, and the set of ______ ______ which make the — **ordered pairs**
6 sentence ______. — **false**

7 The first subset is called the ______ set of the sentence. — **truth, or solution,**

We also found that every point in the plane is
8 associated with an ______ pair of numbers, called the — **ordered**
9 ______ of the point. — **coordinates**

Thus the open sentence sorts the points of the plane into two subsets:

1) the set of all points whose coordinates satisfy the sentence, and

2) all other points.

As before, the first set of points is called the graph of the sentence.

Given the open sentence

$$x - 3y + 6 = 0.$$

The coordinates of the points $(0,2)$, $(-6,0)$, $(9,5)$
10 ______ satisfy the sentence. — **do**
(do, do not)

The coordinates of the points $(2,0)$, $(0,-6)$, $(5,9)$
11 ______ satisfy the given sentence. — **do not**
(do, do not)

The points $(0,2)$, $(-6,0)$, $(9,5)$ belong to the set
12 of points called the ______ of the sentence. — **graph**

13	The points (2,0), (0,-6), (5,9) ______ belong to the graph of this sentence. (do, do not)	

The open sentence "$y = 4$" has been considered as an open sentence in the one variable y. We may, however, also think of this as an open sentence in the two variables x and y written as

$$0 \cdot x + y = 4.$$

	Consider the sentence $y = 4$ as an open sentence in two variables.	
14	Is (3,4) a solution? ______	
15	Is (0,4) a solution? ______	yes
16	Are (-3,4), (2,4), (-10,4) solutions? ______	yes
	What have all these ordered pairs in common?	
17	In each case, the second number, or ordinate, is ______.	4
	Try to visualize the points in the plane associated with these number pairs.	

18	If we consider "$x = -2$" as an open sentence in <u>one</u> variable, the truth set is ______.	{-2}
19	If we consider "$x = -2$" as an open sentence in two variables, we could write $x + \_\_\_ = \_\_\_$.	$x + 0 \cdot y = -2$

20	Some ordered pairs which satisfy this open sentence are [A] (-2,-2), (-5,-2), (3,-2) [B] (-2,-2), (-2,-5), (-2,3)

[redacted]

21	The truth set of $y = 5$ is the set of all ordered pairs whose second number is ____.	5

22 The truth set of $x = 0$ is the set of all ordered pairs whose first number is _____. **0**

What is the graph of an open sentence of the form $2x - 3y - 6 = 0$?

We have discovered that the ordered pairs $(0,-2)$ and $(1,-\frac{4}{3})$ are in the solution set of the equation

$$2x - 3y - 6 = 0.$$

23 We may guess other solutions, such as $(3, \_\_)$. **(3,0)**

It would be easier, however, to determine solutions if we could write an equivalent sentence with y by itself on the left side: Thus,

$$2x - 3y - 6 = 0$$

$$2x - 3y - 6 - (2x - 6) = 0 - (2x - 6)$$

$$-3y = -2x + 6$$

24 $y = \_\_\_\_\_\_.$ **$y = \frac{2}{3}x - 2$**

These are equivalent sentences since we have arrived at the sentence $y = \frac{2}{3}x - 2$ from the sentence $2x - 3y - 6 = 0$ by a series of reversible steps.

The sentence

$$y = \frac{2}{3}x - 2$$

is called the <u>y-form</u> of the sentence

$$2x - 3y - 6 = 0.$$

The sentence $y = \frac{2}{3}x - 2$ may be interpreted in terms of the abscissas and ordinates of the points of the graph. Recall that the x-values correspond to the abscissas of the points on the graph, and the y-values correspond to the ordinates of the points on the graph.

Thus, if

$$y = \frac{2}{3}x - 2,$$

25
26 the ordinate is _____ less than $\frac{2}{3}$ of the _______. **2, abscissa**

	The equation $2x - 3y - 6 = 0$ is equivalent to the equation	
27	$y =$ ______.	$y = \frac{2}{3}x - 2$
	To find the ordinate of a points on the graph of this equation, we would subtract 2 from $\frac{2}{3}$ of the	
28	______.	abscissa
	Thus we might choose abscissas which are multiples of 3.	
29	If the abscissa is 3, the ordinate must be ____, if the sentence $y = \frac{2}{3}x - 2$ is to be true.	0
30	Thus (3,) is a solution of the equation $y = \frac{2}{3}x - 2$. (3,0) is also a solution of the	(3,0)
31	equivalent equation: ______	$2x - 3y - 6 = 0$
32	If the abscissa is -6, the ordinate is ____, if the sentence $y = \frac{2}{3}x - 2$ is true.	-6
33	(-6,) is a solution of the equation $2x - 3y - 6 = 0$.	(-6,-6)

Continuing in this manner, complete the table in Item 34 so that the ordered pairs satisfy the equation $2x - 3y - 6 = 0$, or the equivalent sentence $y = \frac{2}{3}x - 2$.

34

x	-9	-6	-3	0	3	6	9
y		-6		-2	0		

Answer (34): -8, -4, 2, 4

The following ordered pairs are solutions of this equation:

35	(-9,), (-6,-6), (-3,)	(-9,-8), (-3,-4)
36	(,-2), (,0)	(0,-2), (3,0)
37	(6,), (,4)	(6,2), (9,4)

38. Locate the points whose coordinates are given in Items 35, 36, and 37 using the space on the response sheet for Item 38.
Turn to page xviii to check your work. Before continuing, make any necessary corrections.

Have you noticed that the points whose coordinates satisfy the open sentence $y = \frac{2}{3}x - 2$ seem to lie on a straight line? This brings up the following question: If we draw the line which passes through these points, will we find on it every point for which the ordinate is 2 less than $\frac{2}{3}$ of the abscissa? Furthermore, is <u>every point</u> on this line a point whose ordinate is 2 less than $\frac{2}{3}$ of the abscissa?

39	Go back to Item 38 on the response sheet and draw a line connecting the points (-3,-4) and (6,2) and extend the line in both directions as far as the graph paper allows. Then answer the following questions:	
40	Do all of the other points that you located lie on this line? ______ (yes,no)	yes
41	Is there a point on the line whose abscissa is 12? ______ (yes,no)	yes
42	The coordinates of this point are (12,).	(12,6)
43	Is there a point on this line whose ordinate is -5? ______ (yes,no)	yes
44	The coordinates of this point are (,-5).	$(-\frac{9}{2},-5)$
	You found a point with coordinates (12,6) on the line.	
45	Is $6 = \frac{2}{3}(12) - 2$ a true sentence? ______ (yes,no)	yes
	You found another point, $(-\frac{9}{2},-5)$ on the line.	
46	Is $-5 = \frac{2}{3}(-\frac{9}{2}) - 2$ a true sentence? ______ (yes,no)	yes

47	We found coordinates of several points which satisfy the equation $y =$ ______.	$y = \frac{2}{3}x - 2$
48	These points lie on a ______.	line
49	For the points we have found on this line, the ordinate is 2 less than $\frac{2}{3}$ of the ______.	abscissa

325

In fact, every point on this line is a point with ordinate 2 less than $\frac{2}{3}$ of the abscissa.

It is also true that every point with ordinate 2 less than $\frac{2}{3}$ of the abscissa (that is, every point on the graph of $2x - 3y - 6 = 0$) is on the line which you have drawn.

50 When we say that a specified line is the graph of a particular open sentence, we mean:

[A] If an ordered pair of numbers satisfies the sentence, the numbers are coordinates of a point on the line.

[B] If a point is on the line, then its coordinates satisfy the open sentence.

[C] Both of the above.

A specified line is the graph of a particular open sentence with the variables x and y if the coordinates of every point on the line satisfy the sentence and every ordered pair of numbers which satisfies the sentence are the coordinates of a point on the line. Consequently, the correct choice is [C].

We have seen that the graph of the open sentence $2x - 3y - 6 = 0$ is a line. Further, the coordinates of <u>every</u> point on this line satisfy the equation $2x - 3y - 6 = 0$.

Using the procedure developed for the sentence, $2x - 3y - 6 = 0$ we are able to find the graphs of equations like

$$5x + 3y + 11 = 0,$$
$$2x + 5 = 0,$$
$$-6y + 1 = 0.$$

In each case we could conclude that the graph is a line. This suggests the following general statements:

If an open sentence is of the form

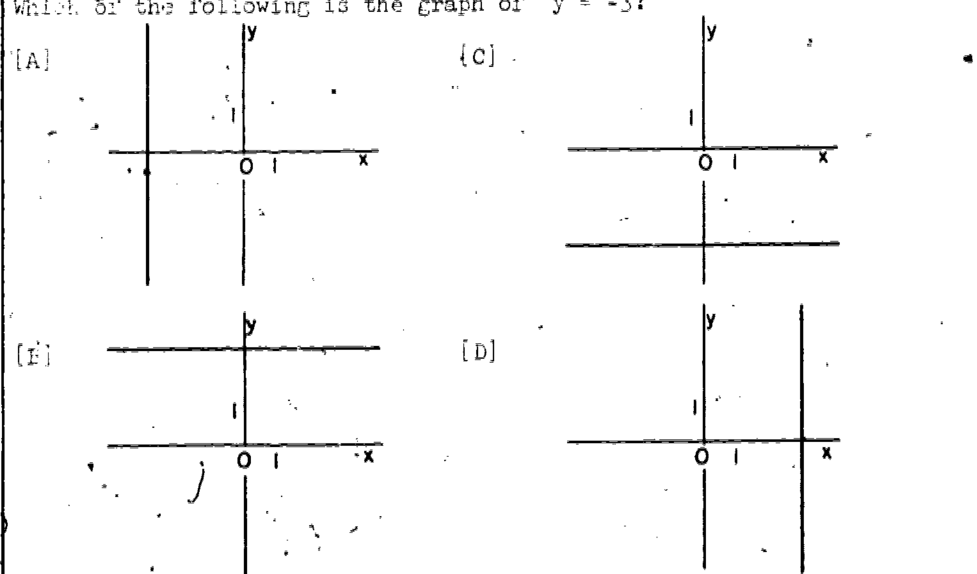
$$Ax + By + C = 0,$$

where A, B and C are real numbers and A and B are not both 0, then the graph of this open sentence is a line. Every line in the plane is the graph of an open sentence of this form.

The graph of an equation of the form $Ax + By + C = 0$	
51 is a ______.	line
In the example $2x - 3y - 6 = 0$,	
52 $A = 2$, $B = -3$, $C =$ ______.	$C = -6$
$3y + 5x + 11 = 0$ is of the form $Ax + By + C = 0$.	
53 In this case $A =$ ___, $B =$ ___, $C =$ ___.	$A=5, B=3, C=11$
The open sentence $2x + 5 = 0$ is also of the form	
54 $Ax + By + C = 0$ with $A =$ ___, $B = 0$, $C =$ ___.	$A=2, B=0, C=5$
In the open sentence $-8y + 1 = 0$,	
55 $A =$ ___, $B =$ ___, $C =$ ___.	$A=0, B=-8, C=1$

Where are all the points in the plane whose ordinates are -3?	
This English sentence can be translated into the open	
56 sentence: $y =$ ______, which is equivalent	$y = -3$
57 to ______ $= 0$.	$y + 3 = 0$
Since this open sentence is of the form $Ax + By + C = 0$, where	
58 $A = 0$, $B =$ ___, $C =$ ___,	$A=0, B=1, C=3$
59 the graph of $y + 3 = 0$ must be a ______.	line

60 Which of the following is the graph of $y = -3$?



Remember that $y = -3$ is an open sentence in two variables such that the ordinate of each point of the graph of this sentence is -3. The correct graph is [C].

Each of the following points, (0,-2), (0,0), (0,3),
61 (0,5), has the value 0 for the ______.

62 All these points are on the ___-axis.

63 An equation whose graph is the y-axis is ______.

The points (-2,0), (0,0), (1,0), (2,0) are all on
64 the ___-axis.

65 An equation for the graph of the x-axis is ______.

66. Consider the equation

$$2y + 5x + 7 = 0.$$

Write this in y-form and make a table of ordered pairs that satisfy this equation. Then graph the equation. Turn to page xx to check your work.

Write the equation

$$x - 4y + 3 = 0$$

67 in y-form. $y =$ ______

68 The ordered pairs (0,) and (,0) satisfy the equation.

69 Locate these points on the coordinate axes and draw the line connecting them, continuing the line to the ends of the paper.

	Do the coordinates of every point on the line satisfy the equation? (Try some values before you answer	
70	this question.) ______ (yes,no)	yes
	If you have drawn the line accurately, the coordinates of <u>every</u> point on the line will be in the truth set of the equation.	
	Write the open sentence for the set of points such that:	
	For each point the abscissa is equal to the opposite	
71	of the ordinate. ______	**x=-y, or y=-x**
	For each point the ordinate is twice the abscissa.	
72	______	**y = 2x**
	For each point the ordinate is the opposite of twice	
73	the abscissa. ______	**y = -2x**

74\. Draw one set of coordinate axes and draw the graph for the set of points for Items 71, 72, and 73. Turn to page xxi to check your work.

75	With reference to one set of coordinate axes, draw the graphs of (a) $y = 3x$, (c) $y = \frac{1}{2}x$ (b) $y = -3x$, (d) $y = -\frac{1}{2}x$ Turn to page xxi to check your graphs. Then complete the following statements:	
76	The graphs are all lines which ______ (do,do not) contain the origin.	do
77	The graphs of (a) and ____ rise from left to right.	(c)
78	The graphs of ____ and ____ descend from left to right.	(b), (d)
79	The graph of (a) is in quadrants I and ____.	I and III
80	The graph of (d) is in quadrants ____ and ____.	II and IV

81 With reference to one set of coordinate axes, draw the graphs of

(a) $y = x + 5$ (c) $y = 2x + 5$

(b) $y = x - 3$ (d) $y = 2x - 3$

Check your graphs with those on page xxii, then complete the following statements:

The graphs of (a) and (b) appear to be a pair of
82 parallel ______. lines

The same appears to be true for the graphs of
83 ______ and ______. (c) and (d)

20-3. Definition of Slope and y-Intercept

Fill in the blanks in the table below so that the ordinate of each ordered pair is equal to the abscissa.

1

x	-6		$-2\frac{1}{2}$		3		6
y		-3		0		5	

See answer below

x	-6	-3	$-2\frac{1}{2}$	0	3	5	6
y	-6	-3	$-2\frac{1}{2}$	0	3	5	6

2 Plot these points and connect the points (-6,-6) and (6,6) with a line. Extend the line in both directions. Check your graph with the graph on page xxii .

Are all of the points in Item 1 on the line through
3 the points (-6,-6) and (6,6)? ______ (yes,no) yes

Does the line pass through the origin (the point
4 (0,0))? ______ yes

All points of the line except (0,0) are in quadrants
5 ______ and ______. I and III

6 The angles formed by the coordinate ______ are bisected by this line. **axes**

7 The open sentence of this graph is ____________. (y = x, y = -x) **y = x**

That is, a point lies on the line if and only if its ordinate is equal to its abscissa.

Label the line "y = x". Continue with Item 8.

Fill in the blanks in the table below so that the ordinate of each pair is the opposite of the abscissa.

8

x	-6		$-4\frac{1}{3}$		2.5		6
y		5.5		0		-4	

See answer below.

x	-6	-5.5	$-4\frac{1}{3}$	0	2.5	4	6
y	6	5.5	$4\frac{1}{3}$	0	-2.5	-4	-6

9 Plot these points using the same coordinate axes as you used for Item 2. Connect the points (-6,6) and (6,-6) with a line and then extend the line in both directions. (Remember that a line extends indefinitely in both directions.) Check your graph with the graph on page xxii. Answer the questions below.

Would the points (20,-20) and (-20,20) be on the line connecting the points (-6,6) and (6,-6)?

10 ______ **yes**

What is an open sentence describing this graph?

11 ______ **y = -x**

This line passes through the origin and also ______ the angles formed by the coordinate axes. (See Item 6.) **bisects**

All points of the line except (6,0) are in

13 quadrants ______ and ______. **II and IV**

Label the graph of y = -x. Note that the graph of

14 y = -x ____________ from left to right. (rises, descends) **descends**

15	The graph of $y = x$ ______ (rises, descends) from left to right.	**rises**
16	On the same coordinate axes used for Items 2 and 9, locate some points such that the ordinate is twice the abscissa. Use a table only if necessary. Connect any two of the points with a line and then extend the line in both directions. Check page xxiii to see if your graph is correct.	
17	Write an open sentence describing this graph. ______ Label the graph of this open sentence.	$y = 2x$
	The graph of $y = 2x$ passes through the point (0,), which is common to $y = x$ and $y = -x$.	(0,0)
	The graph of $y = 2x$ ______ (rises, descends) from left to right.	rises
	Continue using the same graph paper. Locate points such that the ordinate is one-half the abscissa. Connect any two points with a line and continue the line in both directions. Check page xxiv to see if your graph is correct.	
	The open sentence describing this graph is ______.	$y = \frac{1}{2}x$
	Label the graph of $y = \frac{1}{2}x$. This graph passes through the origin.	
	Every point (except (0,0)) on the graph of ______ lies between the graph of $y = \frac{1}{2}x$ and $y = 2x$.	$y = x$
	On the same graph paper, graph the equations $y = -2x$ and $y = -\frac{1}{2}x$. Label the graphs of $y = -2x$, $y = -\frac{1}{2}x$, $y = 0$ and $x = 0$. Turn to page xxiv to check your graphs.	
	All of the graphs contain the point ______.	(0,0)
	The line ______ is between the lines $y = x$ and $y = 0$.	$y = \frac{1}{2}x$
	The line ______ is between the lines $y = -x$ and $y = 0$.	$y = -\frac{1}{2}x$

27	The line $y =$ ___ is between the lines $y = x$ and $x = 0$.	$y = 2x$
28	If you were to graph the equation $y = -6x$ it would be between _______ and $x = 0$.	$y = -x$
29	The graph of $y = .1x$ would be between the graph of $y = x$ and $y =$ ___.	$y = 0$

	Use the graph you have drawn to help answer these questions.	
30	If an equation is in the form $y = kx$, where k is any non-zero real number, then the equation passes through the point (,).	(0,0)
31	If $0 < k < 1$ then the line lies between the lines $y = 0$ and _______.	$y = x$
32	If $k > 1$, the line lies between the lines $y = x$ and _______.	$x = 0$
33	If k is positive the line ___________ (rises, falls) from left to right.	rises
34	If k is _______ the line falls from left to right.	negative
35	If $k = 0$, the graph is the ___-axis.	x-axis
36	What is the y-form of the equation $y = kx$? _______	$y = kx$

In the preceding items we have considered open sentences whose graphs are lines through the origin. The direction of the graph depends on the coefficient of x. As the absolute value of the coefficient increases, the line becomes "steeper".

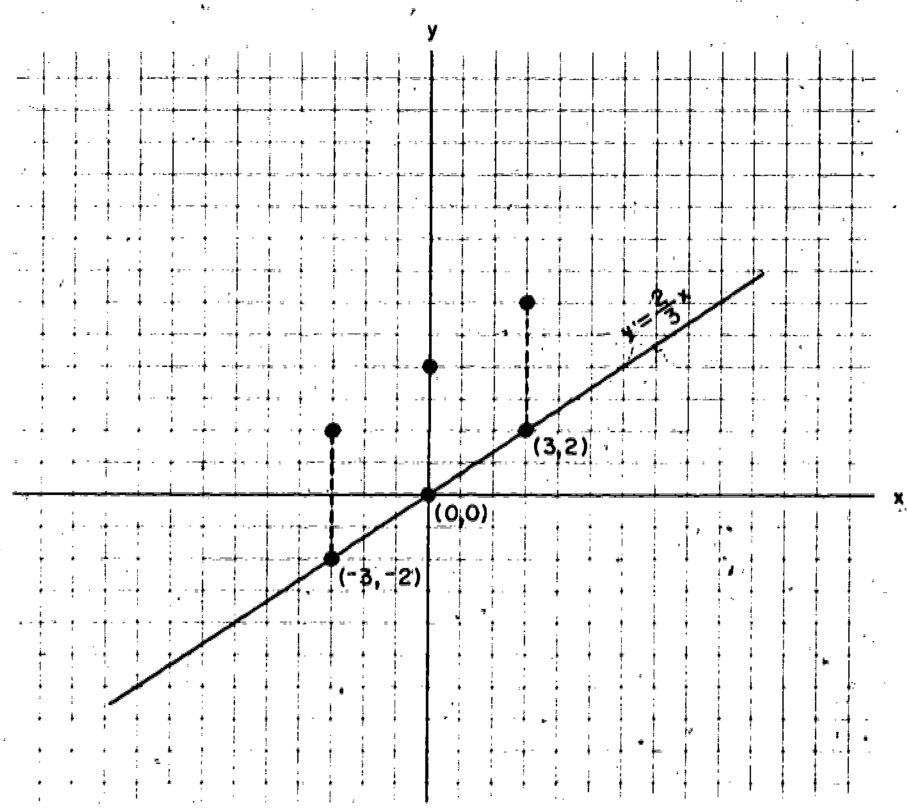
	Now let us consider some lines which do not all contain the origin.	
	In each of the following open sentences (a) $y = \frac{2}{3}x$ (b) $y = \frac{2}{3}x + 4$ (c) $y = \frac{2}{3}x - 3$,	
37	the coefficient of x is _______.	$\frac{2}{3}$

38 The point (0, ___) lies on the graph of the equation $y = \frac{2}{3}x$. (0,0)

For the point on the graph with abscissa 3, the

39 ordinate is $\frac{2}{3}($ ___ $)$ or _____. $\frac{2}{3}(3)$, 2

40 Another point on the line $y = \frac{2}{3}x$ is (-3, ___). (-3,-2)



In the diagram above, the points (-3,-2), (0,0),

41 (3,2) are on the line ______. $y = \frac{2}{3}x$

The point (-3,2) is 4 units above the point

42 (-3, ___). (-3,-2)

The points (-3,2) and (-3,-2) have the same

43 ______________,
(abscissas,ordinates)

44 but different ______________.
(abscissas,ordinates)

45 The points $(0,0)$ and $(\_\_,4)$ have the same abscissas but different ordinates. — $(0,4)$

In fact, if we add 4 to the ordinate of any point on $y = \frac{2}{3}x$, we obtain the ordinate of the corresponding point above it. This is the ordinate of a point on the line $y = \frac{2}{3}x + 4$.

46 The two points have the same ______. — abscissa

To draw the graph of $y = \frac{2}{3}x + 4$, we may add 4 to the ordinates of the points on the graph of $y = \frac{2}{3}x$. Or, we may say that we move the graph of $y = \frac{2}{3}x$ upward 4 units.

Since the point $(0,0)$ is on the line $y = \frac{2}{3}x$, the

47 point $(0, \_\_)$ is on the line $y = \frac{2}{3}x + 4$. — $(0,4)$ $[(0, 0+4)]$

The point $(3,2)$ is on the line $y = \frac{2}{3}x$ and the point

48 $(3, \_\_)$ is on the line $y = \frac{2}{3}x + 4$. — $(3,6)$

The point $(-3,-2)$ is on the line $y = \frac{2}{3}x$ and the

49 point $(-3,2)$ is on the line $y = \frac{2}{3}x + \_\_\_\_$. — $y = \frac{2}{3}x + 4$

50 Graph the lines $y = \frac{2}{3}x$ and $y = \frac{2}{3}x + 4$ using the same coordinate axes. Refer to page xxv to check your graphs.

51 The points $(0, \_\_)$, $(3, \_\_)$, $(-3, \_\_)$ are on the graph of $y = \frac{2}{3}x - 3$. — $(0,-3), (3,-1), (-3,-5)$

In each case, the ordinate of a point on the graph of

52 $y = \frac{2}{3}x - 3$ is _____ less than the ordinate of the corresponding point on the graph of $y = \frac{2}{3}x$. — 3

53 Graph the equation of $y = \frac{2}{3}x - 3$ on the same set of coordinate axes that you used for Item 50. Turn to page xxv to check the graph.

Using the graphs in Item 53, complete the following items.

The graphs of

$$y = \frac{2}{3}x$$

$$y = \frac{2}{3}x + 4$$

$$y = \frac{2}{3}x - 3$$

54 ______ intersect each other. (do, do not) — **do not**

The graph of $y = \frac{2}{3}x$ intersects the y-axis at the
55 point (,). — **(0,0)**

Note that the coordinate of the point of intersection of two lines would be the ordered pair of real numbers which satisfy both equations. This ordered pair is the intersection of the truth sets of the two open sentences.

The graph of $y = \frac{2}{3}x + 4$ intersects the y-axis at
56 the point (,). — **(0,4)**

57 If we add ______ to the ordinate of each point of the — **4**
58 graph of ______ we get the ordinate of the corresponding point of the graph of $y = \frac{2}{3}x + 4$. — **$y = \frac{2}{3}x$**

59 If we subtract ______ from the ordinate of each point of the graph of $y = \frac{2}{3}x$ we get the ordinates — **3**
60 of the corresponding points on the graph of ______. — **$y = \frac{2}{3}x - 3$**

61 The graph of $y = \frac{2}{3}x - 3$ intersects the y-axis at the point (0,). — **(0,-3)**

The point (0,4) is called the y-<u>intercept</u> of the graph of $y = \frac{2}{3}x + 4$. This is the point of intersection of the graph and the y-axis. Since the equation of the y-axis is $x = 0$, the y-intercept is the point of intersection of the lines $y = \frac{2}{3}x + 4$ and $x = 0$.

62 The point (0,4) is the y-______ of the graph of $y = \frac{2}{3}x + 4$. — **y-intercept**

The y-intercept of the graph of $y = \frac{2}{3}x - 3$ is the
63 point (0,). — **(0,-3)**

64 The y-intercept of the graph of $y = \frac{2}{3}x$ is the point (,). **(0,0)**

The number 4 is called the y-intercept number of the equation $y = \frac{2}{3}x + 4$.

65 The y-intercept number of the equation $y = \frac{2}{3}x - 3$ is _____. **-3**

66 The y-______ number of the equation $y = \frac{2}{3}x$ is 0. [illegible]

67 The equation $y = \frac{2}{3}x + 6$ has y-intercept number _____. [illegible]

68 The y-intercept number of the equation $y = mx + b$ is _____. [illegible]

69 The coefficient of x in the equation $y = mx + b$ is _____. [illegible]

Refer back to the graphs in Item 23. The graphs of the equations $y = x$, $y = -x$, $y = 2x$, etc., all have the same y-intercept; but each graph has a different direction. We observed that the direction depended on the coefficient of x.

This leads to the following definition:

The slope of a line is the coefficient of x in the y-form of the equation of the line.

The slope is a number which determines the direction of the line.

70 The slope of the graph of $y = -2x$ is _____. [illegible]

71 The slope of the graph of $y = 3x + 2$ is _____. [illegible]

72 The ______ of the graph of $y = \frac{2}{3}x - 3$ is $\frac{2}{3}$. **slope**

73 What is the slope of the graph of $y = 4$? Since $y = 4$ can be written $y = 0 \cdot x + 4$ the slope is ______. **0**

74 It appears that the slope may be ________, negative, or 0. **positive, negative, or 0**

333

We have defined the slope of a line as the coefficient of x in the y-form of the equation of the line. Thus, the slope of the line $y = mx + b$ is the real number m. Horizontal lines, that is, lines with equations of the form $y = b$ have <u>slope</u> <u>0</u>.

What can we say about the slope for lines with equations of the form $x = a$?

Recall that to find the slope we may begin with the form $Ax + By + C = 0$.

We put this equation in y-form by dividing by B,

75 the coefficient of _____.

However, the equation $x = a$ may be written as

76 $x - a = 0$ and the coefficient of y is _____.

[redacted]

There is no y-form of the equation of the form $x = a$. Lines with equations of this form are <u>vertical lines</u>. The slope is <u>not defined</u> for vertical lines.

77 What can we say about the slope for the line $2x = 4$?

[A] The slope is undefined.

[B] The slope is 0.

[C] The slope is 2.

[redacted]

Find the slope and the y-intercept of the graph of each of the following open sentences. The answers are on page xxvi.

78. $y = x + 207$

79. $y = -x + \frac{7}{2}$

80. $3y = 3x + 11$

81. $3x + y = 1$

82. $2x + 4y - 5 = 0$

83. $7x - 3y + 2 = 0$

84. $3y = 27$

85. $x = -5$

86. $y = mx + b$

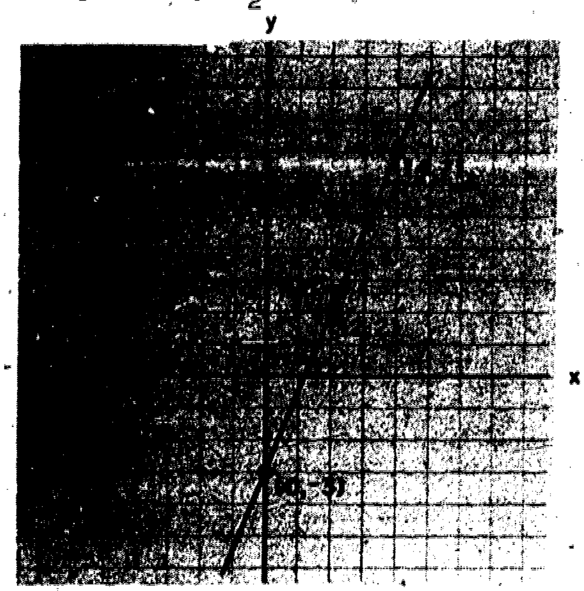
We have seen that if the equation $Ax + By + C = 0$ can be put in y-form, the coefficient of x in the y-form is the slope of the graph and the constant is the y-intercept number.

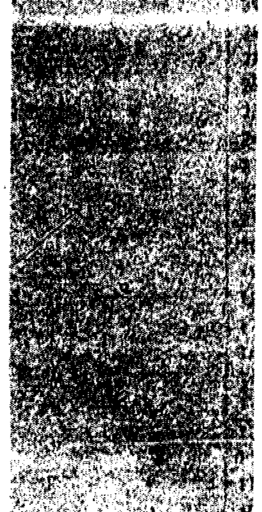
Equations of the form $y = k$ have zero slope. That is, the slope of any horizontal line is 0.

Equations of the form $x = a$ cannot be put in y-form. That is, the slope is undefined for vertical lines.

20-4. Applications of the Slope and Intercept

1 Graph the equation $y = \frac{5}{2}x - 3$. — See answer below.





We have labeled the points $(2,2)$, $(0,3)$, $(4,7)$ in the graph of the line $y = \frac{5}{2}x - 3$.

2 The slope of this line is ______. — $\frac{5}{2}$

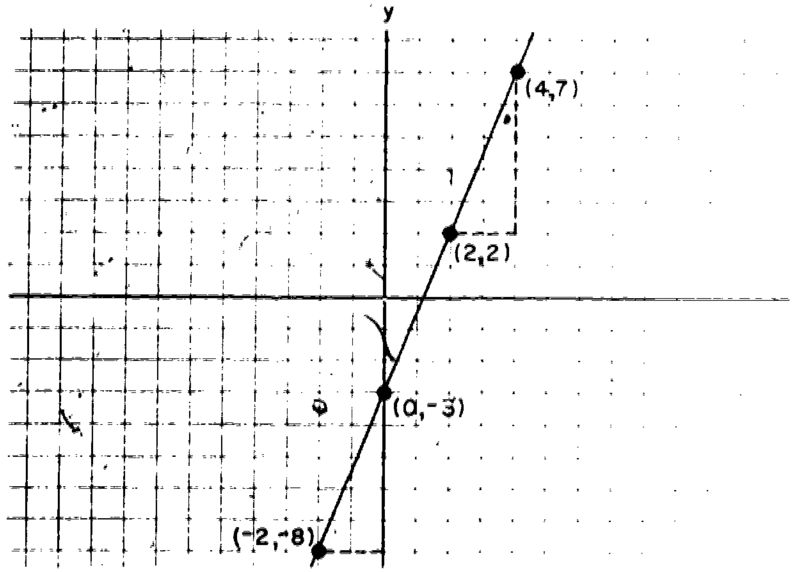
3 The points (,2) and (4,) are on this line. — (2,2) and (4,7)

The ordinates of the points (2,2) and (4,7) are

4 ______ and ______. — 2 and 7

5	The difference of the ordinates is $7 - \_\_\_ = 5$.	$7 - 2$
	We call the difference of the ordinates the <u>vertical change</u> of the line from one point to the other.	
6	The abscissas of these same points are ___ and ___.	2 and 4
7	The difference of these abscissas is $\_\_\_ - 2 = 2$.	$4 - 2$
	The difference of the abscissas is called the <u>horizontal change</u> from one point to another on the line.	
8	The ratio of these differences is $\frac{7-2}{4-2} = \_\_\_$.	$\frac{5}{2}$
9	$\frac{5}{2}$ is the ratio of the differences of the ___ to	ordinates
10	the differences of the ___.	abscissas
	But we know from the definition of slope that $\frac{5}{2}$ is	
11	the ___ of the graph of $y = \frac{5}{2}x - 3$.	slope
	Note the order of the numbers in the ratio $\frac{7-2}{4-2}$.	
	The differences $7 - 2$ and $4 - 2$ came from noting the vertical and horizontal changes <u>from</u> the point with coordinates $(2,2)$ to the point with coordinates	
12	(___ , ___).	$(4,7)$
	Suppose we look at the ratio of the changes <u>from</u> $(4,7)$ to $(2,2)$.	
	Then the first number in the numerator would be 2 and	
13	the first number in the denominator would be ___.	2
14	The ratio would be $\frac{2 - \square}{2 - \square} = \frac{-5}{-2}$.	$\frac{2-7}{2-4} = \frac{-5}{-2}$
15	Since $\frac{-5}{-2} = \_\_\_$, we see that the ratio of the vertical change to the horizontal change from $(4,7)$	$\frac{5}{2}$
16	to $(2,2)$ is the same as the ___ of the vertical change to the horizontal change from $(2,2)$ to $(4,7)$.	ratio

Suppose we had chosen two other points on the graph of $y = \frac{5}{2}x - 3$. Would the ratio of the vertical change to the horizontal change between these two points be $\frac{5}{2}$?



In the above graph we have labeled the points [illegible] (-2,-8).

17 The points (0,) and (-2,) are in the graph of $y = \frac{5}{2}x - 3$. — (0,-3),(-2,-8)

18 The difference between the ordinates [illegible] is ______ = ____. — -3-(-8) = 5

19 The difference between the abscissas _____ and _____ is 2. (Remember to use the first number to correspond to the ordinate -3.) — 0 and -2

20, 21 The ratio of vertical change to horizontal change is _____ which is the ______ of the line. — $\frac{5}{2}$, slope

The above discussion leads to the following general [illegible]:

Theorem 20-4a. For any two points P and Q on a [illegible] line, the ratio of the vertical change to the [illegible] change from P to Q is the slope of the line.

If you choose to omit the proof of this theorem, [illegible]

* **Theorem 20-4a.** For any two points P and Q in a non-vertical line, the ratio of the vertical change to the horizontal change from P to Q is the slope of the line.

Proof:

The equation

$$Ax + By + C = 0$$

*22 is the equation of a _______. — **line**

*23 If $B = 0$ this equation becomes _______. — $\mathbf{Ax + C = 0}$

*24 But this is the equation of a _______ line and the slope of this line is undefined. — **vertical**

We therefore make the restriction, $B \neq 0$.

The y-form of the equation $Ax + By + C = 0$ is

*25 _______. — $\mathbf{y = -\frac{A}{B}x - \frac{C}{B}}$

*26 The slope of this line is _______. — $\mathbf{-\frac{A}{B}}$

If P and Q are points on this line the coordinates

*27 of these points must satisfy the equation _______. — $\mathbf{Ax + By + C = 0}$ [or $\mathbf{y = -\frac{A}{B}x - \frac{C}{B}}$]

Let P have coordinates (a,b) and Q have coordinates (c,d).

The ratio of the vertical change to horizontal change

*28 from P to Q is _______. — $\mathbf{\frac{b - d}{a - c}}$

Since P and Q are on the line

$$Ax + By + C = 0,$$

*29 $Aa + Bb + C = 0$ and also $\underline{A(\quad) + B(\quad) + C = 0}$. — $\mathbf{Ac + Bd + C = 0}$

The sentence: $Aa + Bb + C = 0$ and $Ac + Bd + C = 0$ is equivalent to the sentence:

$$Aa + Bb + C - (Ac + Bd + C) = 0$$

or

*30 $$\underline{A(\quad) + B(\quad) = 0}.$$ — $\mathbf{A(a-c)+B(b-d)=0}$

*31 $A(a - c) + B(b - d) = 0$ is equivalent to $\frac{b - d}{a - c} = -$______. — $\mathbf{\frac{b - d}{a - c} = -\frac{A}{B}}$

(Since P and Q are different points on a non-vertical line, $a \neq c$ so that $a - c \neq 0$.)

*32	Since $-\frac{A}{B}$ is the ______ of the line, and	**slope**
*33	______ is the ratio of vertical change to horizontal change, we have proved that the ratio of the vertical change to the horizontal change from P to Q is the slope of the line.	$\frac{\mathbf{b - d}}{\mathbf{a - c}}$

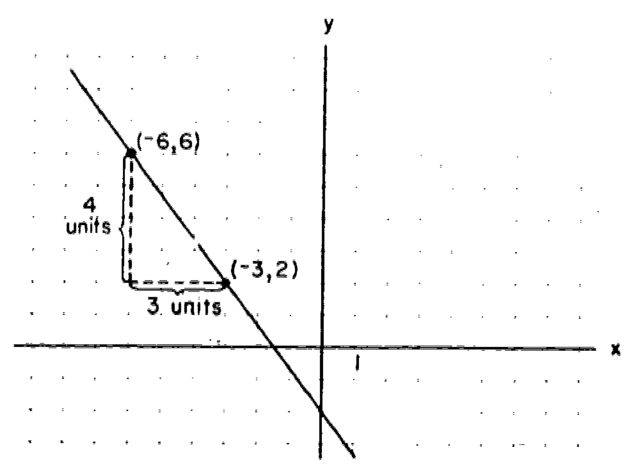
Using the theorem we have just stated, we can always find the slope of a [illegible] if we are given the coordinates of two points on the line.

34	If the coordinates of two points on a line have the same ordinate the line is ______________. (vertical, horizontal)	**horizontal**
35	If the coordinates of two points on a line have the same abscissas the line is ______________. (vertical, horizontal)	**vertical**

Find the slope of the line through each of the following pairs of points. Compare your answers with those on page xxvi .

36. (-7,-[illegible]) and ([illegible],[illegible])	39. ([illegible],[illegible]) and ([illegible],0)
37. (-7,[illegible]) and ([illegible],[illegible])	40. (0,0) and (-[illegible],-2)
38. ([illegible],-[illegible]) and (-[illegible],[illegible]0)	41. (0,0) and (-7,4)

Examine the following graph:



 314

Since the points (-6,6) and (-3,2) are on this line, we know the slope is

42 $\dfrac{2 - 6}{-3 - (-6)}$ or ____.

We could check this by counting the squares. That is,
43 from (-3,2) to (-6,6) there are ____ units in the vertical change.

44 There are 3 units in the ______ change.

[illegible] horizontal [illegible]

45 What is the equation of this line? We could write this equation in y-form if we knew:

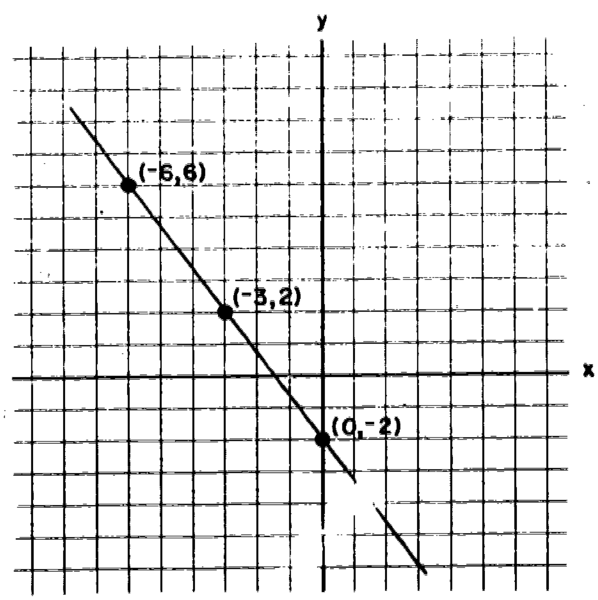
[A] the slope

[B] the y-intercept number

[C] both the slope and the y-intercept number

[illegible] equation is $y = mx + b$. The coefficient of [illegible] constant is the y-intercept [illegible] of these we can write the equation of [illegible] the equation $y = kx$ has y-intercept [illegible] is also in the general form $y = mx$ [illegible] answer is [C].

By examining the above graph we find that the y-intercept is (0,-2).



The slope of this line is $-\frac{4}{3}$. (See Item 42.)

46 The y-intercept number of this line is ______, since -2

47 the line intersects the y-axis at ______. $(0,-2)$

48 The equation of this line is $y =$ ______. $y = -\frac{4}{3}x - 2$, $[y = -\frac{4}{3}x + (-2)]$

The equation of a line parallel to this line and

49 containing the point $(0,12)$ is $y =$ ______. $y = -\frac{4}{3}x + 12$

A line containing the point $(0,6)$ and parallel to the line whose equation is $y = \frac{3}{5}x - 2$ has the

50 equation $y =$ ______. $y = \frac{3}{5}x + 6$

What is the equation of a line with slope $-\frac{5}{6}$ and

51 y-intercept number -3? ______ $y = -\frac{5}{6}x - 3$

What is the open sentence of a line containing the points $(4,11)$ and $(2,4)$, and having y-intercept $(0,-3)$?

Since the line passes through the points $(2,4)$ and $(0,-3)$, the slope of the line is

52 $$\frac{4 - (\quad)}{2 - (\quad)} = \underline{\qquad}.$$ $\frac{4 - (-3)}{2 - 0} = \frac{7}{2}$.

53 The equation of the line is $y =$ ______. $y = \frac{7}{2}x - 3$

The ordered pair $(4,11)$ also satisfies the equation

54 $y = \frac{7}{2}x - 3$ since $11 = \frac{7}{2}\cdot 4 - 3$ is ______ (true, false).   true

The equation of the line can always be found from the

55 slope and y-______.   y-intercept

If you want more practice on this type of example, do Items 56-65; otherwise skip to the material following Item 65.

	To write an equation of the line containing $(-1,0)$	
56	and $(0,3)$, we need to know the ______ of the line	**slope**
57	and the y-________.	**y-intercept**
58	The slope of the line is $\frac{3 - \ }{0 - (\ )}$ = ______.	$\frac{3-0}{0-(-1)} = 3$
59	The y-intercept is the point ______.	$(0,3)$
	An equation of the line containing $(-1,0)$ and $(0,3)$	
60	is ______.	$y = 3x + 3$

	Write an equation of the line through each of the following pairs of points.	
61	$(0,3)$ and $(-5,2)$ ______	$y = \frac{1}{5}x + 3$
62	$(5,8)$ and $(0,-4)$ ______	$y = \frac{12}{5}x - 4$
63	$(0,-2)$ and $(-3,-7)$ ______	$y = \frac{5}{3}x - 2$
64	$(3,3)$ and $(0,3)$ ______	$y = 3$
65	$(3,3)$ and $(3,0)$ ______	$x = 3$

If we are given the slope and the y-intercept of a line, can we graph the line?

	Suppose a line has slope $-\frac{2}{3}$ and y-intercept number, 6.	
66	The equation of this line is $y =$ ______	$y = -\frac{2}{3}x + 6$
	One way to graph this would be to find some ordered pairs in the truth set.	
67	We may also draw the graph by first locating the y-intercept ______ and then trying to locate another point whose coordinates would satisfy the given equation.	$(0,6)$
68	Since the slope is $-\frac{2}{3}$ we know that between any two points on this graph, the ratio of ______ change to	**vertical**
69	______ change is $-\frac{2}{3}$.	**horizontal**

70 If the vertical change between the points is [illegible], the horizontal change [illegible] is ____. — 3

71 If the horizontal change is [illegible], the vertical change would be ____. — 4

Thus, if we have a [illegible] point on the line with slope $-\frac{2}{3}$, we can always find another point by going either [illegible]

72 ______. — right

If we go [illegible] units to the left, we would go [illegible] units

73 ______ (up, down). — up

If we go [illegible] units to the right, we would go [illegible] units

74 ______. — down

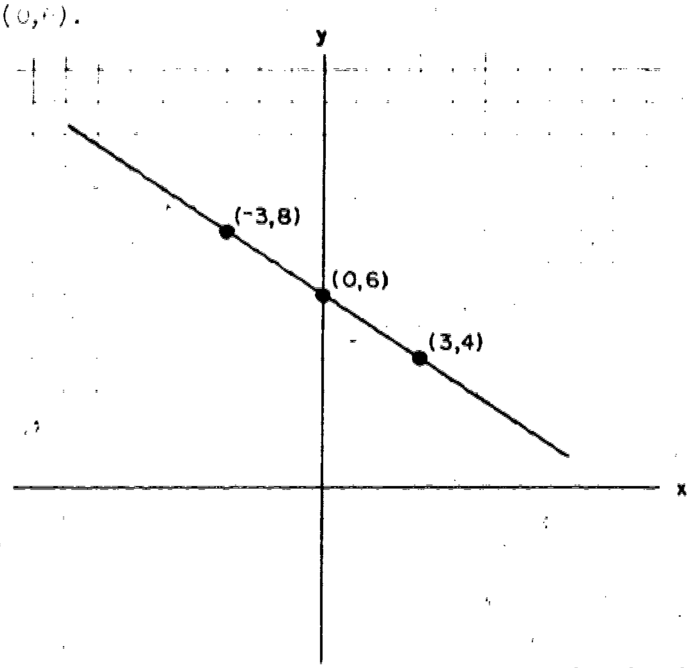
If we go [illegible] units to the right and [illegible] units down

75 from (0,6) we get the point (___ , ___). — (3,4)

If we go [illegible]

76 (0,6) [illegible] (___ , ___). — (-3,8)

77 Locate the points [illegible], [illegible] and [illegible]; then draw the line with slope $-\frac{2}{3}$ and y-intercept (0,6). — See answer below.



Find the equation and graph each of the following lines. Compare with the answers on pages xxvi, xxvii, xviii.

[illegible]. A line with slope 5/[illegible] and y-intercept number is 0.

[illegible]. A line through the point $(-6,-3)$ and with slope undefined.

[illegible]. A line through the point $(0,1)$ and with slope $-\frac{1}{2}$.

[illegible]. A line through the point $(-6,-3)$ and with slope $\frac{5}{6}$. (Remember that we can find a second point on this line by going 6 units to the right and then 5 units up.)

What is the slope of the line containing the points $(-3,2)$ and $(3,-4)$?

82 The slope is ____. — -1

Suppose we label another point on this line with the coordinates (x,y).

83 Then the slope is also $\dfrac{\square - 2}{\square - (-3)}$. — $\dfrac{y-2}{x-(-3)}$

Since the slope is -1, and the point (x,y) is different from the point $(-3,2)$, we may write

84 $\dfrac{y-2}{x+3} =$ ____. — $\dfrac{y-2}{x+3} = -1$

This leads to the equation

85 $y - 2 = -1(x + 3)$ or $y =$ ____. — $y = -x - 1$

Check that this is the equation of the line by showing that the ordered pairs $(-3,2)$ and $(3,-4)$ satisfy this equation.

What is the equation of the line through the points $(-[illegible],3)$ and $(-5,3)$?

86 The slope of this line is ____. — 0

87 The equation is ____. — $y = 3$

If the line contains the points $(-3,3)$ and $(-3,5)$,

88 the equation is ____. This is a vertical line. — $x = -3$

89	If the line contains the points $(4,2)$ and $(-3,1)$ the slope of the line is _____.	$\frac{1}{7}$
90	If (x,y) is another point on this line the slope is $\frac{y - \square}{x - \square}$.	$\frac{y-1}{x-(-3)}$ or $\frac{y-2}{x-4}$
91	Since (x,y) is different from $(4,2)$ we can write $\frac{y - \square}{x - \square} = \frac{1}{7}$	$\frac{y-2}{x-4} = \frac{1}{7}$
92	The equation of the line is $y - 2 = \frac{1}{7}(x + 3)$ or $y =$ ________.	$y = \frac{1}{7}x + \frac{10}{7}$
93	(x,y) is also different from $(-3,1)$, so we can also write $\frac{y - \square}{x - \square} = \frac{1}{7}$.	$\frac{y-1}{x-(-3)} = \frac{1}{7}$
94	The equation of the line is $y - 2 = \frac{1}{7}(x - 4)$ or $y =$ ________.	$y = \frac{1}{7}x + \frac{10}{7}$
	Check that the points $(4,2)$ and $(-3,1)$ are on this line!	

	Consider a non-vertical line.	
	We have been writing the equation of such a line in either of the forms:	
95	$Ax + By + C = 0$, where we insist $B \neq$ _____ or	$B \neq 0$
96	$y = mx + b$, where m is the ______ of the line.	slope
	Examining the second of these forms, we see that the right-hand side "$mx + b$" is a polynomial in the	
97	variable _____.	x
	If $m \neq 0$ the degree of the polynomial $mx + b$	
98	is ______.	one

	Any polynomial of the first degree is called a <u>linear</u> polynomial.	
99	Thus, $4x + 2$, $-2x + 7$, and $\frac{1}{2}x$ are all ______ polynomials in x.	linear

	If k and n are real numbers, $k \neq 0$, then	
100	$kx + n$ is a _______ _______ in x.	linear polynomial
	The adjective "linear" is applied to such a polynomial	
101	since the graph of $y = kx + n$ is a _______.	line
	We sometimes refer to the graph of the polynomial	
	$kx + n$, rather than to the graph of the equation	
102	y __________.	$y = kx + n$
	Thus "the graph of the linear polynomial $x + 1$" means	
103	"the graph of the open sentence ________ $x + 1$."	$y = x + 1$

With reference to a single set of coordinate axes, draw the graphs of the following linear polynomials.

Compare your graphs with those on page xxviii.

104. $x + 1$ 106. $\frac{1}{2}x - 2$

105. $2x +$ [illegible] 107. $-\frac{[illegible]}{[illegible]}x + \frac{[illegible]}{2}$

20-9. Summary and Review

There is a one-to-one correspondence between the set of ordered pairs of real numbers and the set of points in the real number plane.

The graph of the equation

$$Ax + By + C = 0$$

is a line if A and B are not both 0.

If $A = 0$, the line is horizontal.

If $B = 0$, the line is vertical.

If $B \neq 0$, we may write the equation $Ax + By + C = 0$ in the form

$$y = mx + b.$$

The form $y = mx + b$ is called the __y-form of the equation of the line__.

In the y-form, the coefficient of x is called the slope of the line and the constant term is the __y-intercept number__.

The point of intersection of the line and the y-axis is called the y-intercept.

The slope is also the ratio of vertical change to horizontal change between two points on the line.

The slope of a horizontal line is 0.

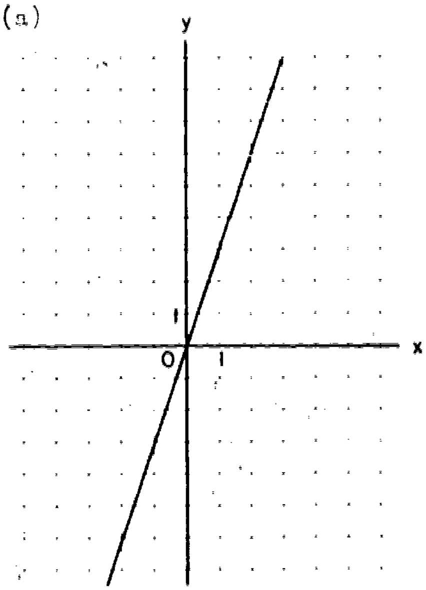
The slope of a vertical line is not defined.

Review Problems.

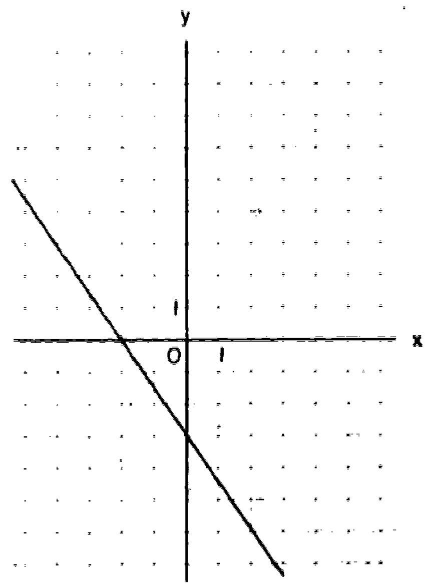
Answers for the review problems are on pages xxix, xxx, xxxi, xxxii, xxxiii.

1. For each of the following graphs, write an open sentence.

(a)



(b)



2. Draw the graph of each of the following open sentences.

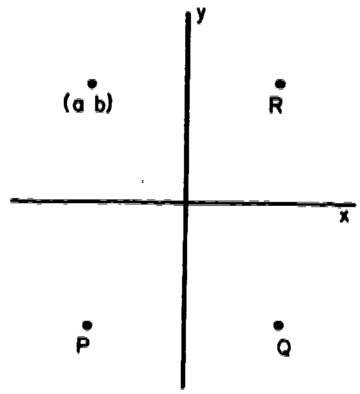
(a) $y = \frac{x}{2} + 7$

(b) $x + y = 0$

(c) $3y - 12 = 0$

(d) $2x + 5y - 6 = 0$

3. The point (a,b) at the right is in the second quadrant.



   (a) Is a positive?

   (b) Is b positive?

   (c) If the coordinates of P, Q, and R have the same absolute values as the abscissa and ordinate of (a,b), state the coordinates of P, Q, and R in terms of a and b.

   (d) If (c,d) is a point in the third quadrant, in which quadrant is the point $(c,-d)$? The point $(-c,d)$? The point $(-c,-d)$?

4. Draw the graph of "$y = 3x + 4$". On the same set of axes graph:

   (a) $y = 3(-x) + 4$

   (b) $y = -(3x + 4)$

   (c) $y = (3x + 4) - 5$

   (d) $y = 3(x - 2) + 4$

   Which of these lines are parallel?

5. (a) With reference to one set of axes, draw the graphs of:

$$2x + y - 5 = 0$$
$$6x + 3y - 15 = 0$$

   What is true about these two graphs? Now look at the equations; how could you get the second equation from the first?

   (b) What is true of the graphs of

$$Ax + By + C = 0$$

   and

$$kAx + kBy + kC = 0$$

   for any non-zero k?

6. (a) With reference to one set of axes, draw the graphs of

$$[illegible]x - 4y - 12 = 0,$$

$$[illegible]$$

What is true about these two graphs? What is true about the coefficients of x and y in these equations?

(b) [illegible]

$$[illegible]$$

and

$$[illegible]$$

for any non-zero [illegible] $\neq 0$.

7. Write the equations of the lines passing through the following pairs of points.

(a) (0,0) and ([illegible])

(b) ([illegible] ,- [illegible]) and ([illegible])

(c) ([illegible]) and ([illegible])

(d) ([illegible] ,- [illegible]) and ([illegible] ,- [illegible])

8. Suppose [illegible] (a,b) is moved to the point (a - [illegible], b + [illegible]).

(a) To what points [illegible] points move?
([illegible]), ([illegible]), ([illegible]), ([illegible]), ([illegible] ,0)

(b) To what point does (a, b - [illegible]) go?

(c) Which points [illegible] themselves.

9. Consider a rectangle whose length is [illegible] units greater than its width w.

(a) Write an expression in w for the perimeter of the rectangle. Is this a linear expression in w?

(b) Write an expression in w for the area of the rectangle. Is this a linear expression in w?

*10. Consider a circle of diameter d.

(a) Write an expression in d for the circumference of the circle. Is this expression linear in d? What happens to the circumference if the diameter is doubled? Halved? If c is the circumference, what can you say about the ratio $\frac{c}{d}$? How does the value of c change when the value of d is changed.

(b) Write an expression in d for the area of the circle. Is this expression linear in d? If A is the area of the circle, what can you say about the ratio $\frac{A}{d}$? What about the ratio $\frac{A}{d^2}$?

353

Chapter 21

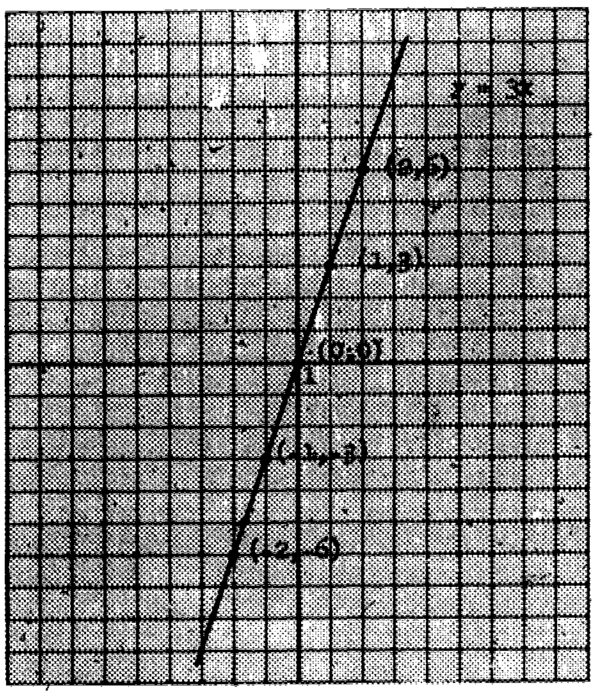
GRAPHS OF OTHER OPEN SENTENCES IN TWO VARIABLES

21-1. Graphs of Inequalities

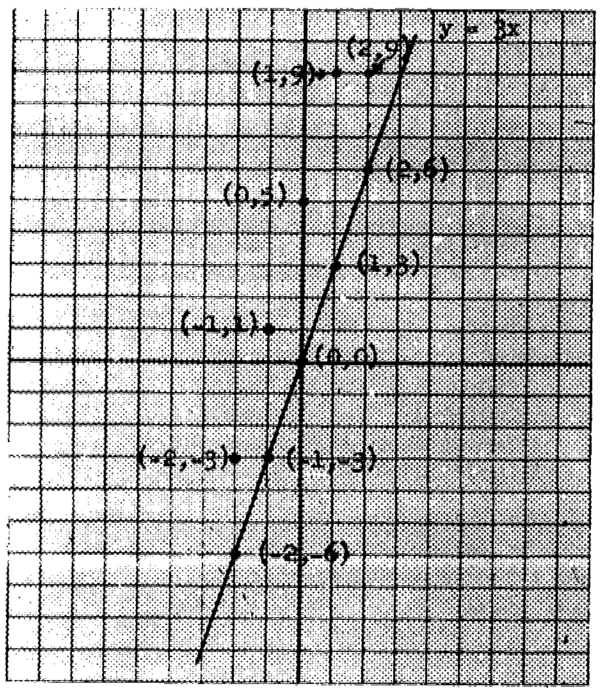
The graph of those points such that the ordinate of each
1 point is 3 times the abscissa is a ______. line

2 The equation of this graph is y = ____. $y = 3x$

3 Graph the equation $y = 3x$ and label the points $(-2,-6)$, $(-1,-3)$, $(0,0)$, $(1,3)$, $(2,6)$. [see answer below]



4. On the same set of coordinate axes as that used in Item 3, locate and label the points (-2,-3), (-1,1), (0,5), (1,9), (2,9).	[See answer below]
5. The point (-2,) is on the line y = 3x.	(-2,-6)
6. Do all of the points in Item 4 lie on one line? ______ (yes,no)	no
7. Let us think of one point as "corresponding" to another if they have the same abscissa. Thus, the point in Item 4 corresponding to (-2,-6) is (-2,).	(-2,-3)
8. Does the point (-2,-3) lie above the corresponding point (-2,-6)? ______ (yes,no)	yes



How can we express the relationship between the ordinates of the points on the line y = 3x and the ordinates of corresponding points above the line y = 3x?

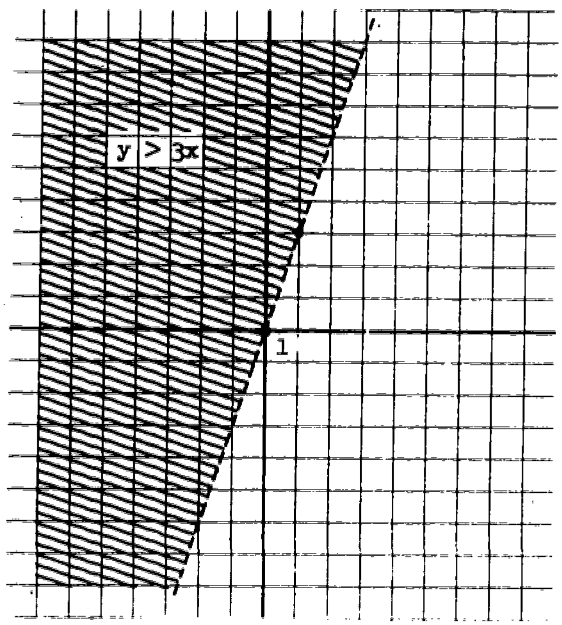
9	The ordinate of a point above the line ________________ (is greater than, is less than) the ordinate of the corresponding point on the line.	is greater than
10	The ordinate of each point on the line is ______ times the abscissa.	3
11	This is expressed by the open sentence $y =$ ____.	$y = 3x$
12	For the points above the line, the ordinate of each point is more than 3 times the ______.	abscissa
13	This may be expressed by the open sentence y ____ $3x$. (<,=,>)	$y > 3x$

Do the coordinates of every point above the line $y = 3x$ satisfy the open sentence $y > 3x$?

14	For example, (6,18) is on the line $y = 3x$. A corresponding point above the line with abscissa 6 would be (6, __) (Write an ordinate satisfying the condition.)	[Any ordered pair with abscissa 6 and ordinate greater than 18 is correct.]
15	For any point with abscissa 6, and ordinate greater than 18, is $y > 3x$ a true sentence? ____ (yes, no) (Try any ordinate greater than 18 and you will see that this is true.)	yes

The open sentence $y > 3x$ is satisfied by every point above the line $y = 3x$. That is, the truth set of the sentence $y > 3x$ consists of all the points above the line $y = 3x$. The graph of the sentence $y > 3x$ is the graph of the truth set. Thus, the graph of $y > 3x$ is the set of all points in the plane which lie <u>above</u> the line $y = 3x$.

We shall indicate the graph of the open sentence $y > 3x$ by shading the portion of the plane containing all of the points in the truth set:

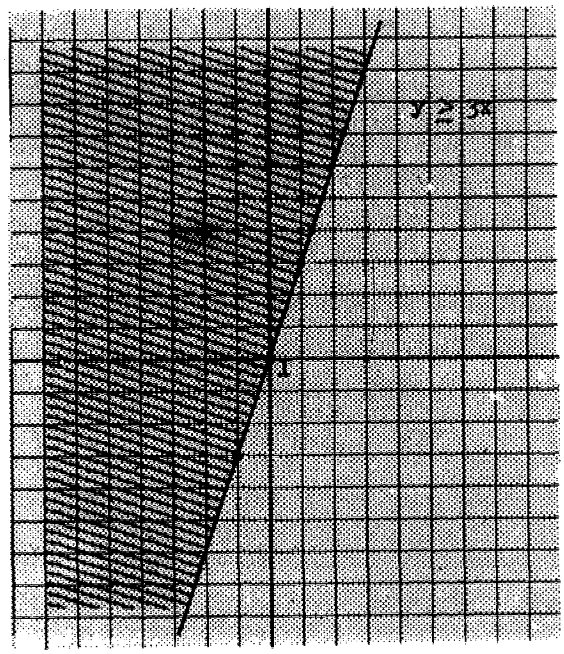


The points on the line $y = 3x$ are <u>not</u> in the truth set of the sentence $y > 3x$. We draw a dashed line to show that the points on the line $y = 3x$ are <u>not</u> included in the graph of the truth set. The dashed line is the <u>boundary</u> of the region of the plane all of whose points satisfy the open sentence $y > 3x$.

16	How would we graph the truth set of the sentence $y \geq 3x$? The truth set of this sentence is the ______________ (union, intersection) of the truth sets of the sentences $y > 3x$ and $y = 3x$.	union
17	Thus, the graph would consist of the region of the plane ______ the line $y = 3x$ <u>and</u> the	above
18	line <u>$y =$ ____</u>.	$y = 3x$

35J

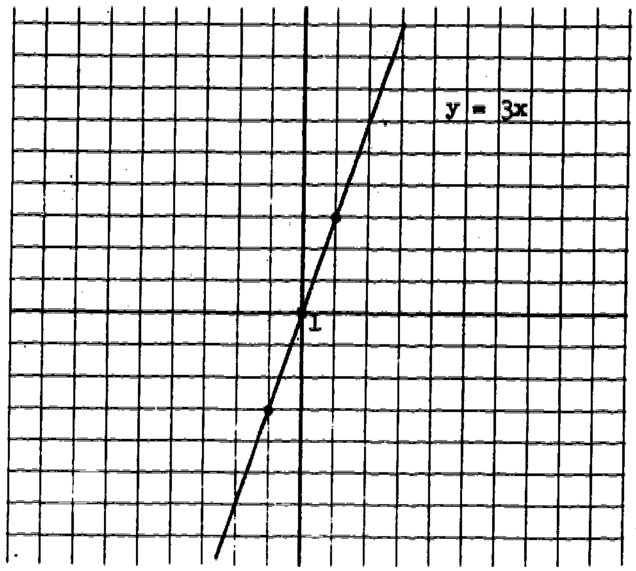
19 The graph of $y \geq 3x$ is



Graph of $y \geq 3x$

Note that the graph of $y = 3x$ is drawn as part of the graph as a heavy line, since the points on the line $y = 3x$ <u>are</u> in the truth set of $y \geq 3x$.

Consider the graph of $y = 3x$:



The graph of this equation is a line. The points of the plane are above the line, below the line or on the line. We see that the line separates the plane into two half-planes.

20	The graph of $y < 3x$ is the set of points ______ the line $y = 3x$.	below
21	The graph of y ___ $3x$ is the half-plane such that for every point the ordinate is less than three times	$y < 3x$
22	the ______.	abscissa
23	The graph of $y \leq 3x$ is the ______ containing all	region
24	points such that every ordinate is less than or ______ to three times the abscissa.	equal to

Using a separate set of coordinate axes for each, draw the graph of:

25. $y = x$
26. $y = x + 2$
27. $y > x + 2$
28. $y > x + 2$

Compare your graphs with those on page xxxiv.

Draw the graphs of the following open sentences on separate coordinate axes.

29. $2x - 7y = 14$ 31. $2x - 7y < 14$

30. $2x - 7y > 14$ 32. $2x - 7y \geq 14$

Turn to page xxxv to check your answer.

The line which determines the half-plane in the graph

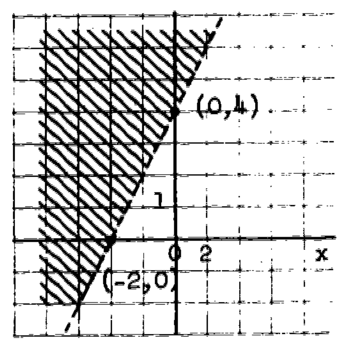
33 below has a slope of _______

34 and its y-intercept is _______.

35 The line is the graph of $y =$ _______.

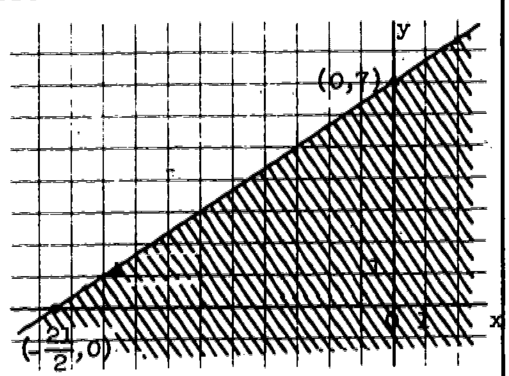
The open sentence which describes

36 the shaded region is _______.



The open sentence describing the shaded region of this graph is

37 y _______.



38 On a sheet of graph paper and with reference to different se
draw the graphs of the following and determine which of them
points (0,10) and (-5,4).

R. $y \geq \frac{3}{4}x - 1$ S. $y < \frac{2}{3}x + 7$ T. $y \leq 2$

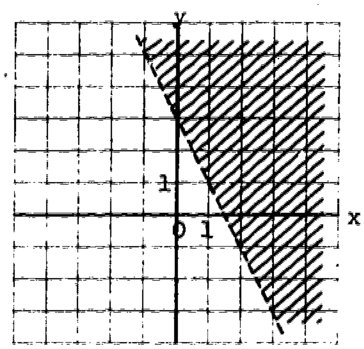
[A] R only [B] R and S [C] S and

39 Examine each of the following graphs and open sentences and match each open sentence with its graph. Which list below associates the sentence and its graph correctly?

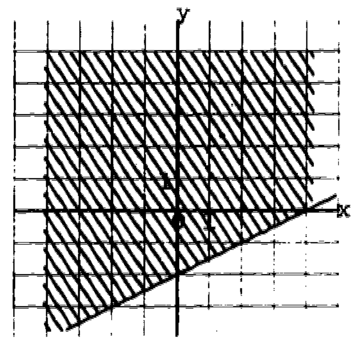
G. $2x + y > 3$ J. $x - 2y \leq 4$

H. $x + 2y \geq 4$ K. $2x - y \leq 3$

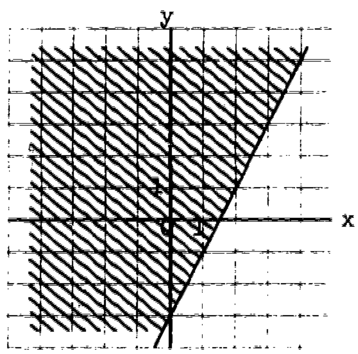
L.



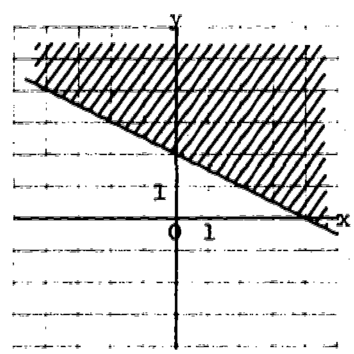
N.



M.



R.



[A] G, L; H, R; J, N; K, M

[B] G, L; H, N; J, R; K, M

[C] neither [A] nor [B]

[redacted]

If you had trouble with Item 39, continue with Item 40. If not, go on to the next sentence.

You might find it helpful to write the open sentences in Item 39 in a form similar to the y-form of the equation of a line.

40 $2x + y > 3$ may be written as $y >$ ______. — $y > -2x + 3$

41 $x + 2y \geq 4$ may be written as y ______. — $y \geq -\frac{1}{2}x + 2$

42 $x - 2y \leq 4$ may be written as y ______. — $y \geq \frac{1}{2}x - 2$

43 $2x - y \leq 3$ may be written as y ______. — $y \geq 2x - 3$

Go back to Item 39 to see that the [A] is the correct choice.

21-2. Graphs of Open Sentences Involving Absolute Value

Consider the following as open sentences in two variables:

$$x = 3$$
$$|x| = 3.$$

1 The graph of $x = 3$ is a ______. — line

What does the sentence $|x| = 3$ mean when we think of this as a sentence in two variables?

We recall that $|x| = 3$ is equivalent to the compound

2 sentence $x = 3$ or ______. — $x = 3$ or $-x = 3$ [$x = 3$ or $x = -3$]

That is, the sentence is satisfied by all ordered pairs of real numbers which satisfy the sentence $x = 3$, and also by the ordered pairs which satisfy the sentence

3 $x =$ ______. — $x = -3$

The truth set of the open sentence $|x| = 3$ is

4 the ______ of the truth set of $x = 3$ and the truth set of $x = -3$. — union

5 The graph of the sentence $x = 3$ is a ______ line three units to the right of the line $x = 0$; that is, three units to the right of the y-axis. — vertical

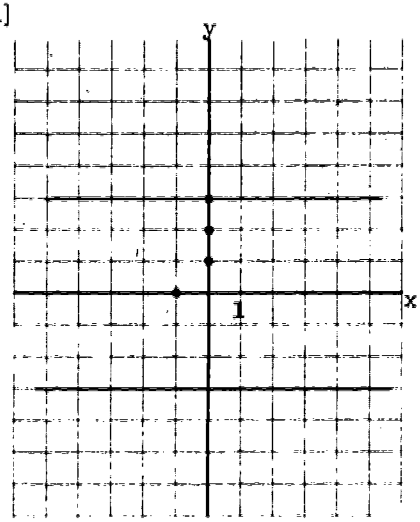
6 The graph of the sentence $x = -3$ is a vertical line three units to the ______ of the y-axis.

7 The graph of $|x| = 3$ is the union of the graph of ______ and the graph of ______.

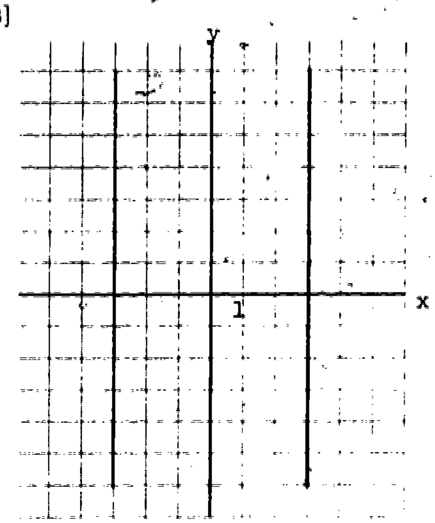
[redacted]

8 Which of the following is the graph of $|x| = 3$?

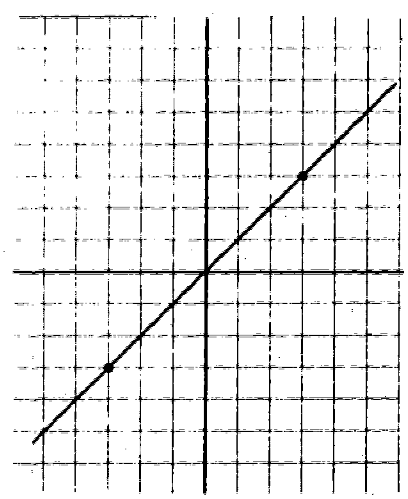
[A]



[B]



[C]



[redacted]

9 The equation $|y| = 2$ is equivalent to the compound sentence ______ or ______.

10 The graph of $y = 2$ is a ______ line two units above the x-axis.

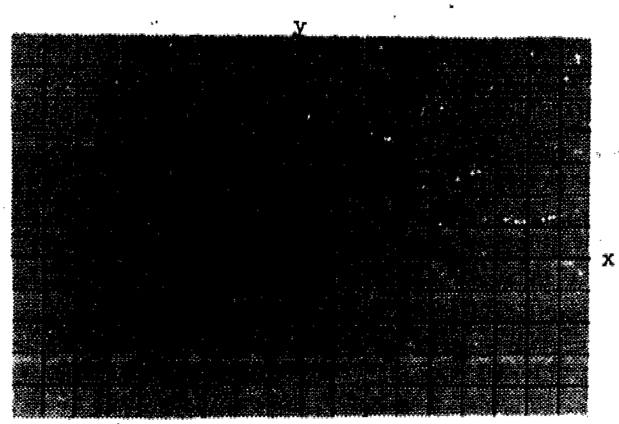
The graph of $y = -2$ is a horizontal line two units below the ______.

11

12 The graph of $|y| = 2$ is the ______ of the graphs

13 of $y = 2$ and ______.

14 The graph of $|y| = 2$ is



Graph the following open sentences. Turn to page xxxv to cl graphs.

15. $|x| = 5$

16. $|y| = 1$

17. $|x| = 7$

The equation $|x| = k$ is equivalent to the compound sentenc $x = -k$" for any real number k. Thus, the graph of the equatior always the union of the graphs of $x = k$ and $x = -k$.

What is the graph of the equation $|x| = 0$?

$|x| = 0$ is equivalent to the compound sentence

18 "$x = 0$ or ______" but "$x = 0$" and "$x = -0$" are equivalent sentences since $-0 = 0$.

The graph of $|x| = 0$ is the union of the graphs of $x = 0$ and $x = -0$; which, in this case, is the <u>single</u>

19 line ______ 0.

20 The graph of $x = 0$ is the ____-axis.

$x = -0$

$x = 0$

y-axis

The graph of $|y| = 0$ is the same as the graph of the

21 open sentence $y =$ ____.

Did you remember that "$y = 0$ or $y = -0$" is equivalent to $y = 0$?

22 The graph of $|y| = 0$ is the ____-axis.

$y = 0$

x-axis

---

23 What is the graph of $|x| = -1$?

[A] The pair of lines $x = -1$ and $x = 1$.

[B] The whole number plane since the truth set is the set of all real numbers.

[C] The graph contains <u>no</u> points, since the truth set is the empty set.

The correct choice is [C]. Since the absolute value of a real number is <u>always</u> non-negative, there can be no ordered pairs of real numbers satisfying the open sentence $|x| = -1$.

What is the graph of $|x - 3| = 2$?

$|x - 3| = 2$ is equivalent to the compound sentence

24 $x - 3 = 2$ or ______.

25 Thus, the truth set of $|x - 3| = 2$ is the ______ of the truth sets of $x - 3 = 2$ and $x - 3 = -2$.

The truth set of $x - 3 = 2$ is the same as the truth

26 set of x ______.

27 The graph of $x = 5$ is a ______ line, 5 units to

28 the right of the ____-axis.

$-(x - 3) = 2$
$[x - 3 = -2]$

union

$x = 5$

vertical line

y-axis

The truth set of $x - 3 = -2$ is the same as the truth

29 set of x ______.

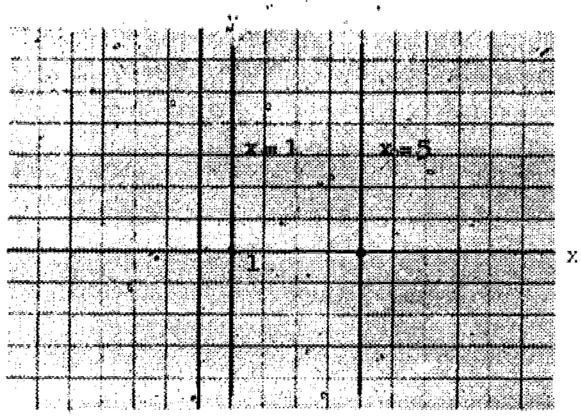
30 The graph of $x = 1$ is a vertical line ______ unit(s)

31 to the right of the ______-axis.

Thus, the graph of $|x - 3| = 2$ is the pair of lines

32 x ______ and $x =$ ______.

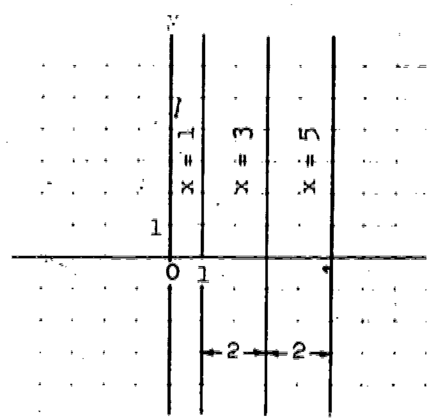
33 The graph of $|x - 3| = 2$ is:



Recall that on the number line "$|x - 3| = 2$" is interpreted as: "distance between x and 3 is 2."

Extending this notion to the number plane, we can refer to the graph $|x - 3| = 2$ as a pair of vertical lines such that the distance between each of the lines and the line $x = 3$ is 2.

The figure shows that the distance between the line $x = 1$ and the line $x = 3$ is 2, and the distance between the line $x = 3$ and the line $x = 5$ is 2.



34. Describe the graph of $|x + 3| = 1$. The sentence $|x + 3| = 1$ can be written in the form $|x - (\quad)| = 1$.

Using the same reasoning as above we can now state our problem in the following way:

Where are the points in the plane such that the distance between them and the line ______ is 1 ? 35

36 These points are on the lines $x = -4$ and ______.

The graph of $|x + 3| = 1$ is a pair of

37 ______________ lines.
(vertical, horizontal)

We could have used the definition of absolute value to describe the graph.

$|x + 3| = 1$ is equivalent to

38 $x + 3 = 1$ or $-(\quad\quad) = $ ______.

An equivalent sentence is

39 $x = -2$ or $x =$ ______.

Thus, the truth set of $|x + 3| = 1$ is the same as the truth set of the compound open sentence

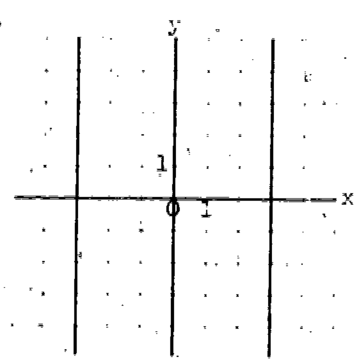
40 ______ or ______.

[redacted]

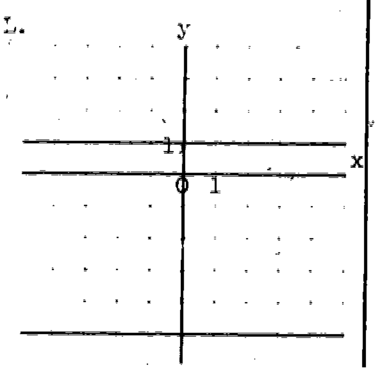
From the above discussion we see that we arrive at the same result using the notion of "distance between" or using the properties of absolute value.

Given below are four graphs and some open sentences. For each open sentence, either indicate the associated graph, or write "none" if the graph of the sentence is not one of those given.

K.



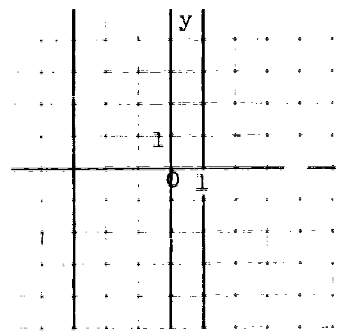
L.



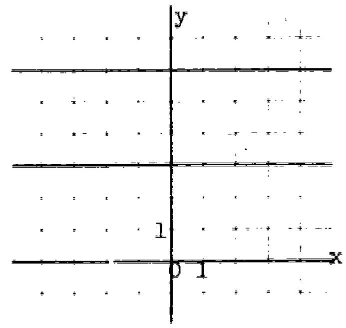
(continued)

[redacted]

M.



N.



41 $|x| = 3$ ______

42 $|x + 2| = 3$ ______

43 $|y| = 3$ ______

44 $y + 2 = 3$ or $y + 2 = -3$ ______

45 $|x + 1| = 2$ ______

46 $|y + 2| = 3$ ______

47 $|y - 0| = 3$ ______

How would we find the graph of the open sentence $|x| > 3$?

Using the properties of absolute values, we know that $|x| > 3$ is equivalent to "$x > 3$ or $-x > 3$"; that is,

48 to "$x > 3$ or $x <$ ___".

The graph of $x > 3$ is the half-plane to the right of

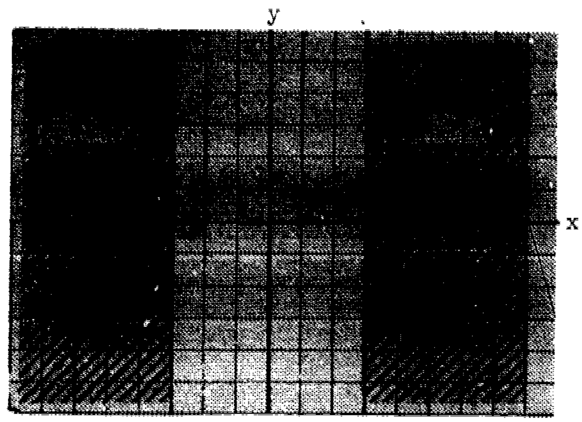
49 the line $x =$ ______.

The graph of $x < -3$ is the half-plane to the left of

50 the line $x =$ ______.

51 Since the truth set of $|x| > 3$ is the ______ of the truth sets of $x > 3$ and $x < -3$, the graph of

52 $|x| > 3$ will be the following:



Draw the graphs of the following open sentences, each with reference to a different set of axes:

53. $|x| > 2$ 54. $|x| \geq 2$

Turn to page xxvii to check your work.

How would we find the graph of $|x| < 5$?

The open sentence $|x| < 5$ is equivalent to "$x > -5$ and $x < 5$".

Another way of expressing this is "$x \geq 0$ and $x < 5$ or

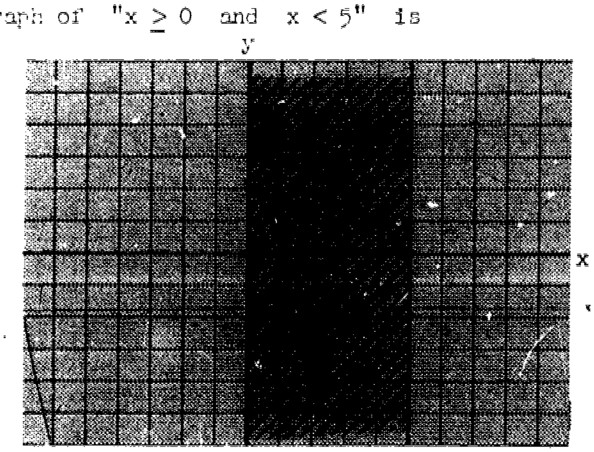
55 $x < 0$ and $x >$ ______". $x > -5$

The graph of "$x \geq 0$ and $x < 5$" is the region of the

56 plane between the lines $x =$ ______ and $x =$ ______ $x = 0$, $x = 5$

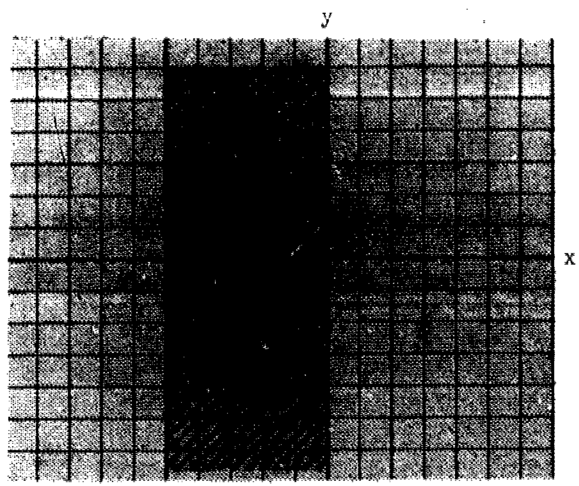
and the line $x = 0$.

57 The graph of "$x \geq 0$ and $x < 5$" is [See answer below]



The graph of "$x < 0$ and $x > -5$" is the region of the plane between the lines $x = 0$ and x ______.

The graph of "$x < 0$ and $x > -5$" is



Note that we have indicated that the y-axis is not part of the graph by making it a dashed line.

60 Since the graph of $|x| < 5$ is the ______ of the graphs of "$x \geq 0$ and $x < 5$" and "$x < 0$ and $x > -5$", it is the region of the plane between the

61 lines $x =$ ______ and $x =$ ______.

62 Graph $|x| < 5$ and turn to page xxxvi to check your graph.

$x = -5$

[See answer below]

union

$x = -5$, $x = 5$

Consider the open sentence $y = |x|$. Since the absolute value of a number is defined for all real numbers we know that there must be a value of the variable y for every value of the variable x. Let us draw the graph of this open sentence.

For any real number, the value of the variable y in

63 the sentence $y = |x|$ is non-______.

If x has the value 0, the value of the variable

64 y is ______.

non-negative

0

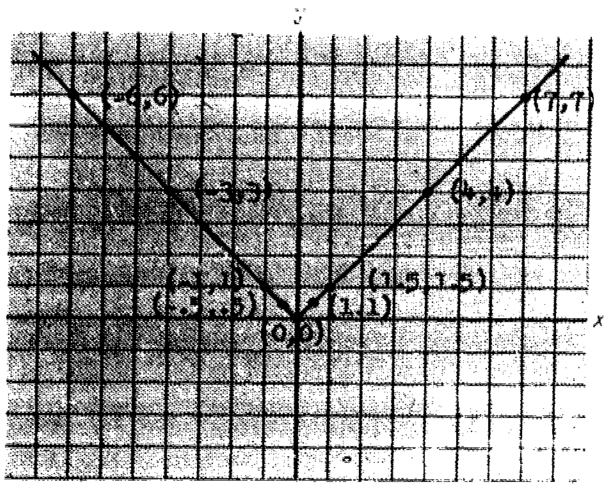
373

65	Complete the following table of values for the open sentence $y = |x|$.	[See answer below]

x	-6	-[illegible]	-[illegible]	-.[illegible]	0	[illegible]	[illegible]	[illegible]	[illegible]
y			[illegible]	[illegible]	[illegible]	[illegible]			

x	-	-[illegible]	-[illegible]	-.[illegible]	[illegible]		.	[illegible]	
y				.			.		

66 Locate these points and graph the [illegible] [See answer below]



Be sure your graph of $y = |x|$ [illegible]

Then answer the following questions:

67 The graph of $y = |x|$ is never [illegible] the ______-axis. — x-axis

68 It contains the point (,). [illegible] — (0,0)

(0,0), the graph of $y = |x|$ [illegible]

69 ______ and ______. — I and II

The graph of $y = |x|$ is formed by two rays [illegible] the

70 point (0,). — (0,0)

71 The angle formed by the two rays is a ______ [illegible]. — right

Complete each of the following:

For the points of the graph of $y = [illegible]|x|$, each ordinate is twice the ordinate of the corresponding point of

72 y = ______. — $y = |x|$

The graph of $y = 2|x|$ contains the point $(0, \_\_)$. | $(0,0)$

The graph of $y = \frac{1}{2}|x|$ each ordinate is one-half as large as the ordinate of the corresponding point of ______. | $y = |x|$

The graph of $y = \frac{1}{2}|x|$ also contains the point $(0, \_\_)$. | $(0,0)$

For the points of the graph of $y = -|x|$, each ordinate is the ______ of the ordinate of the corresponding point of $y = |x|$. | opposite

All of the points except $(0,0)$ of the graph of $y = -|x|$ lie in quadrants ______ and ______. | III and IV

With reference to a separate set of axes, draw the graph of each of the following:

78. $y = 2|x|$ 80. $y = -|x|$

79. $y = \frac{1}{2}|x|$ 81. $y = -2|x|$

Turn to page xxxvii to check your work.

82 Given the open sentence $x = |y|$.

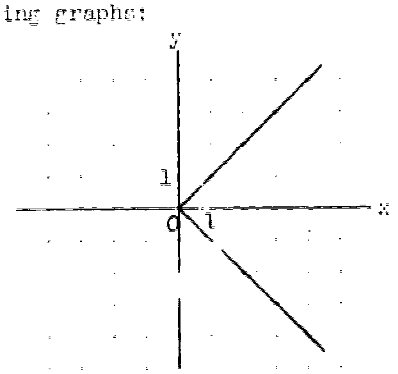
The graph of this open sentence contains the origin and extends

[A] above the x-axis.

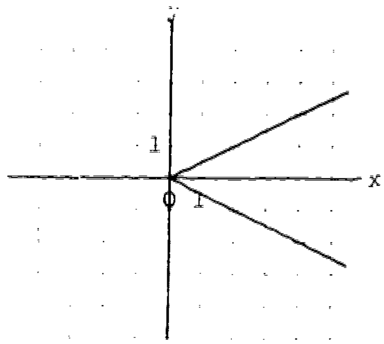
[B] to the right of the y-axis.

$x = |y|$ means that all abscissas are non-negative for any real values of the ordinate y. The correct choice is [B].

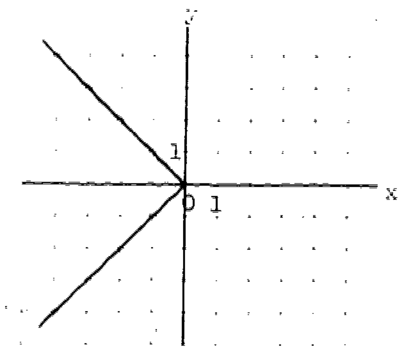
Write the open sentence associated with each of the following graphs:

83 

______ | $x = |y|$

84 

$x =$ ______

85 

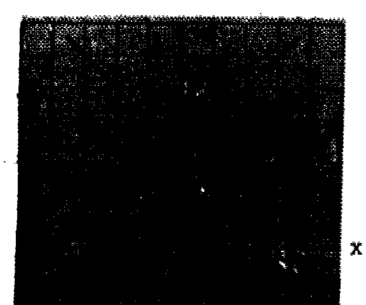
$x =$ ______

86 The graph of $y = |x|$ is formed by two rays which start at (0,0) and form a ______ angle.

87 The graph of $y = |x| + 3$ is formed by two rays which start at (0,).

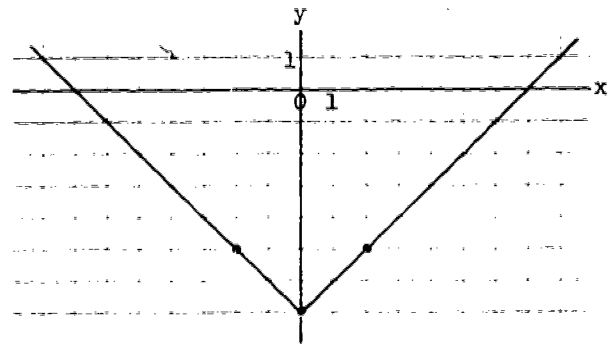
88 Each ordinate is ______ (how many) units more than the corresponding ordinate of $y = |x|$.

89 The graph of $y = |x| + 3$ is

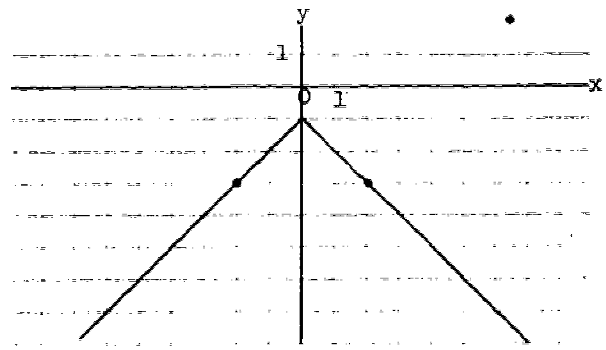


Write the open sentence associated with the following graphs.

(a)



(b)



90 (a) ______

91 (b) ______

[redacted]

92 On scratch paper sketch the graph of each of the following pairs of open sentences with reference to a separate set of axes. Then answer the question below.

1. $y = \|x\|$	2. $y = 2\|x\|$	3. $y = -3\|x\|$
$y = \|x\| + 2$	$y = 2\|x\| + 3$	$y = -3\|x\| + 1$

How is the graph of the sentence $y = k|x| + b$ (where b is a positive real number) related to the graph of $y = k|x|$?

[A] The graph of $y = k|x| + b$ (where b is a positive number) is the result of sliding the graph of $y = k|x|$ vertically b units up.

[B] The graph of $y = k|x| + b$ (where b is a positive number) is th result of sliding the graph of $y = k|x|$ horizontally b units to the right.

Note that the graphs of $y = k|x|$ all contain the origin, whereas in each case the graphs of $y = k|x| + b$ meets the y-axis at the point $(0,b)$. The correct choice is [A].

93 How may the graph of $y = -k|x|$ be obtained from the graph of $y = k|x|$?

[A] By revolving the graph of $y = k|x|$ one half-revolution about the x-axis.

[B] By revolving the graph of $y = k|x|$ one half-revolution about the y-axis.

Since the ordinates $k|x|$ and $-k|x|$ are opposites of one another, the correct choice is [A].

With reference to the same set of axes draw the graph of
94 $y = |x|$ and the graph of $y = |x - 3|$. Check with the graph on page xxxvii.

The graph of $y = |x - 3|$ can be obtained from the
95 graph of $y = |x|$ by sliding it ________ units to
(how many)
96 the ________.
(left, right)

3

right

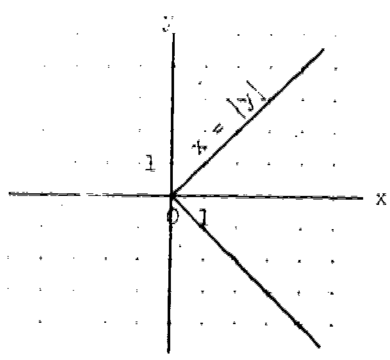
Draw the graph of each pair of open sentences with reference to a separate set of axes.

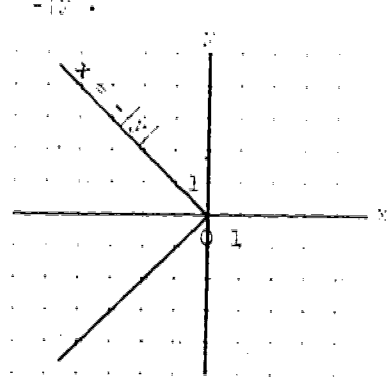
97. $y = |x|$ 98. $y = -|x|$ 99. $y = \frac{1}{2}|x|$

$y = [illegible]|x + [illegible]|$ $y = -|x + 5|$ $y = \frac{1}{2}|x - 1|$

Turn to page xxxvii to check your work.

If the graph of $x = |y|$ is revolved one half-revolution about the y-axis, the resulting graph is that of $x = -|y|$.





37[illegible]

The graph of $y = -2\|x + 3\|$ may be obtained from the graph of $y = 2\|x\|$ by sliding it ______ (how many) units to the left and revolving it about the ______-axis.	3
	x-axis
We can arrive at the same result by revolving $y = 2\|x\|$ first about the x-axis and then sliding it 3 units to the ______.	left

What does the graph of $|x| + |y| = 5$ look like? Let us make a table first. Suppose we start with the intercepts. Let $y = 0$, and get the possible values of x which make the sentence true. Then let $x = 0$, and get the possible y values. Now fill in the rest of the table.

x	-[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
$\|x\|$	5					0	0					5
$\|y\|$	0					5	5					0
y	0					5	-5					0

When you have completed this table, turn to page xxxviii and check. Make any corrections before continuing.

The entries in the first row of the table can be paired with those in the fourth row to form a set of ordered pairs, (x,y), which satisfy the open sentence

$$|x| + |y| = 5.$$

Draw the graph of $|x| + |y| = [illegible]$.

Turn to page xxxviii to check your work.

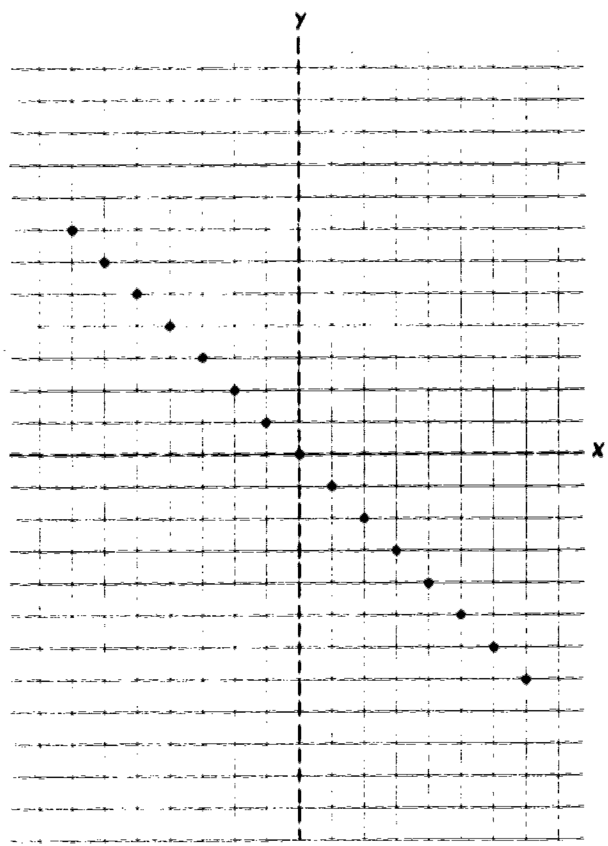
373

*21-3. Graphs of Open Sentences Involving Integers Only

Thus far we have considered open sentences in two variables where the domain of the variables has been the set of all real numbers. What happens to the graph of an open sentence in two variables if we restrict the domain of the variables to be the set of integers?

Before we study the graphs, let us agree to use dashed lines to indicate the coordinate axes in a plane in which we consider only points with integer coordinates.

Consider the following graph:



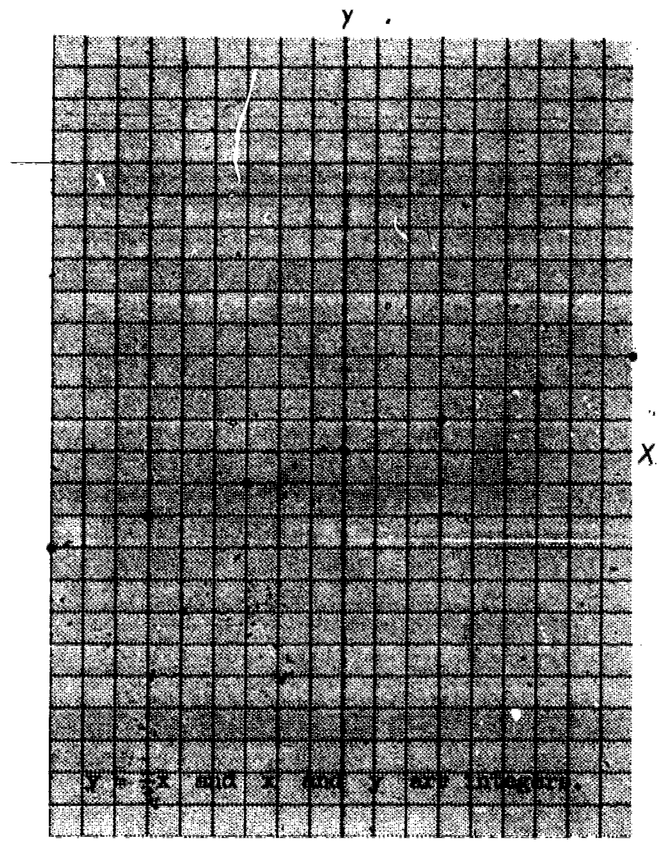
The points on this graph have coordinates which

1 are ______.

Some of the points on this graph are

2 (0,), (1,), (-1,).

3	The ordinate of each point is the _____ of the abscissa.	opposite
4	An open sentence which describes the condition in Item 3 is $y =$ _____.	$y = -x$
5	The open sentence which completely describes this graph is $y = -x$ and x and y are _____	integers
6	and $-10 \le x \le$ _____.	$-10 < x < 10$
7	Consider the following situation: Let x and y be integers such that the value of y is $\frac{1}{3}$ the value of x. Plot the graph and write the open sentence.	[See answer below]



381

8 What integers would be in the domain of the open sentence y

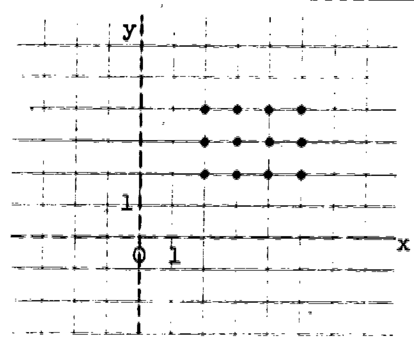
[A] x may have all integral values and y may have all int

[B] x may have values which are multiples of 3 and y ma integral values.

[C] x may have all integral values and y may have values multiples of 3.

[redacted]

Next, consider the set of twelve points in this graph. Find an open sentence which describes this set.



9 We can write immediately that x and y are ______.

10 The smallest value for the abscissas is ______.

11 The largest value for the abscissa is ______.

In the same way, the smallest and largest values for

12 the ordinates are ______ and ______, respectively.

This suggests the compound open sentence:

13 "$1 < x <$ ______ and $1 < y <$ ______,

14 and x and y are ______."

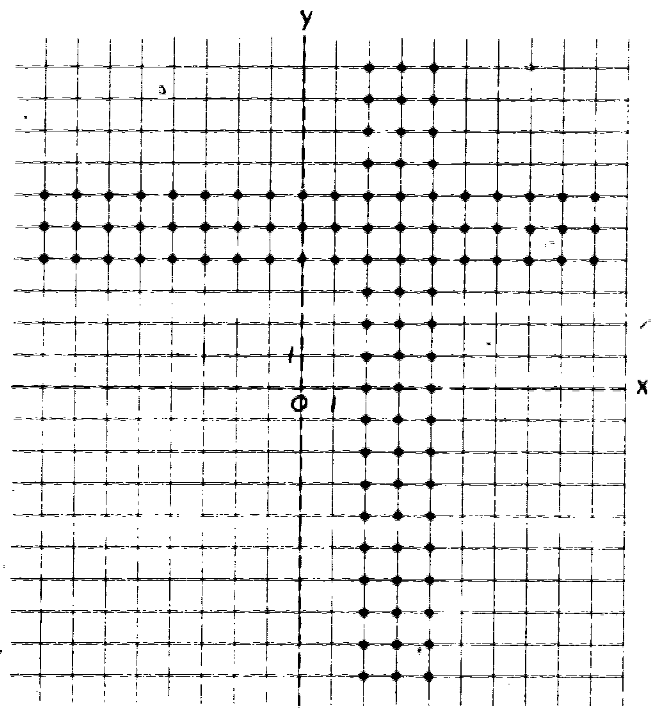
The same set of points could be described by:

15 "______ $\leq x \leq$ ______ and ______ $\leq y \leq$ ______,

and x and y are integers."

Notice that the connective for the compound sentence is and. The truth set is the intersection of the truth set of the sen $1 < x < 6$ and the truth set of the sentence $1 < y < 5$.

382



Find an open sentence which will describe the infinite set of points on the graph above.

An open sentence which describes the three horizontal rows of dots is

16 rows of dots is $\_\_\_ < y < \_\_\_$, and x and y are integers. $3 < y < 7$

An open sentence which describes the three columns of

17 dots is $\_\_\_ < \_\_\_ < \_\_\_$, and x and y are integers. $1 < x < 5$

The graph includes the points which belong to the horizontal rows, the columns, or both.

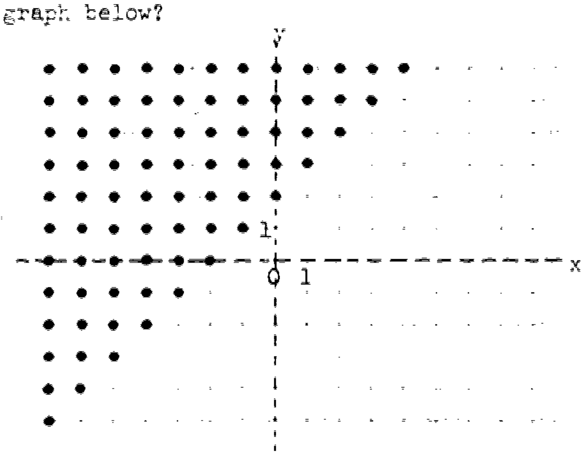
An open sentence describing the total set of points is,

18 therefore, $1 < x < 5$ _____ $3 < y < 7$, and x and y are integers. or

Another open sentence describing this set of points is

19 $|x - 3| < 2$ or $|y - 5|$ _____ and x and y are integers. $|y - 5| < 2$

20 What is an open sentence that describes the infinite set of points indicated by the graph below?



[A] $y > x + 2$, and x and y are integers.

[B] $y \geq x - 2$, and x and y are integers.

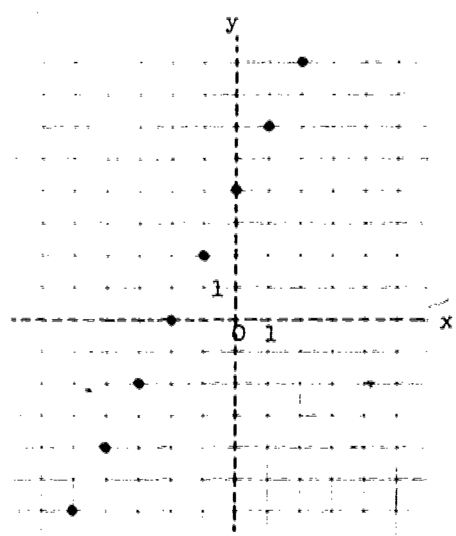
[C] $y \geq x + 2$, and x and y are integers.

The correct choice is [C].

21 Below are three open sentences and three graphs. Match each sentence with the corresponding graph.

R. $y = \frac{x}{2}$, $-6 < x < 6$, and x and y are integers.

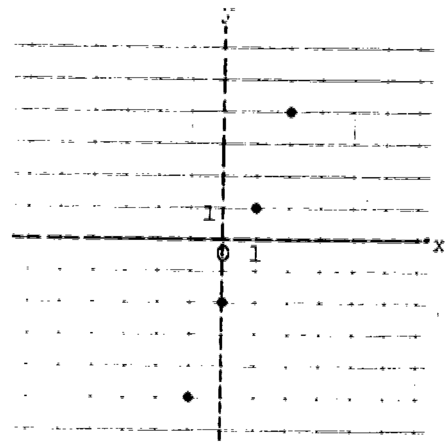
L.



(continued)

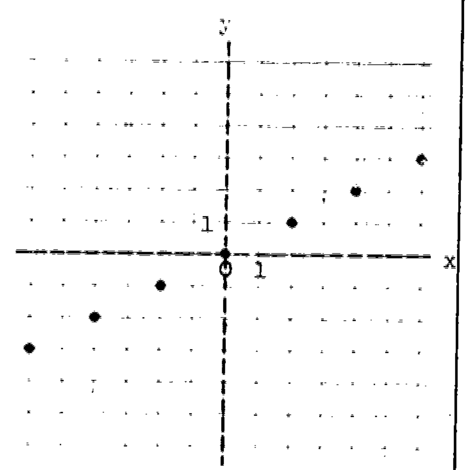
S. $y = 3x - 2$, and
x and y are integers.

M.



T. $y = 2x + 4$, and
x and y are integers.

L.



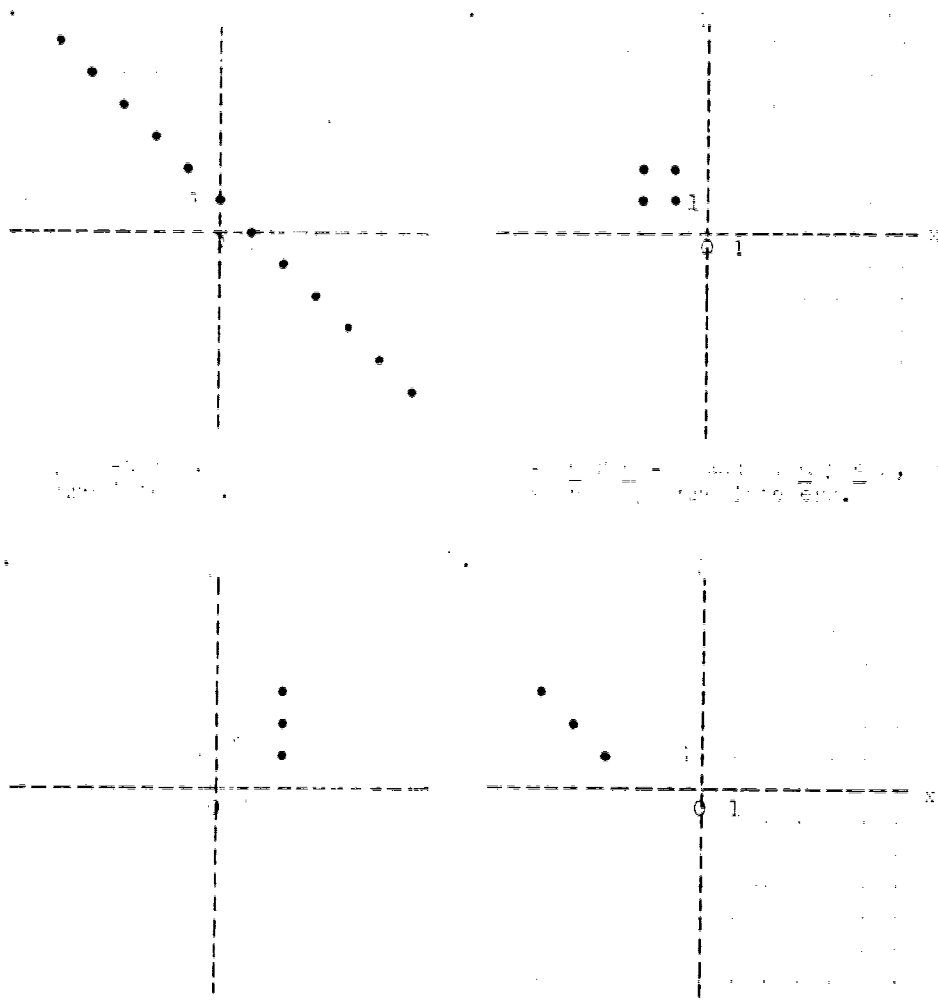
[A] R, L; S, N; T, M

[B] R, N; S, M; T, L

[C] R, M; S, N; T, L

The correct choice is [B].

2. [illegible]



[illegible]

[illegible] $-x$ [illegible] and $-6 < x < -2$ and [illegible] and x and y are integers.

[illegible], U.

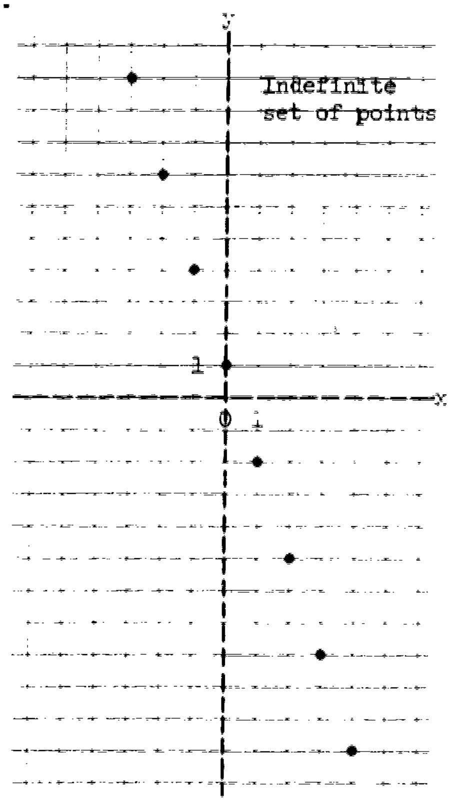
For graph S there would have to be further restrictions on the variable to limit the vertical column to three points, namely $1 \leq y \leq 3$. In the case of graph U for no one of the three points $(-3,1)$, $(-4,2)$, $(-5,3)$, is it true that the ordinate is two more than the opposite of the abscissa.

A correct open sentence for U is $y = -x - 2$, $-6 < x < -2$ and $0 < y < 4$, and x and y are integers.

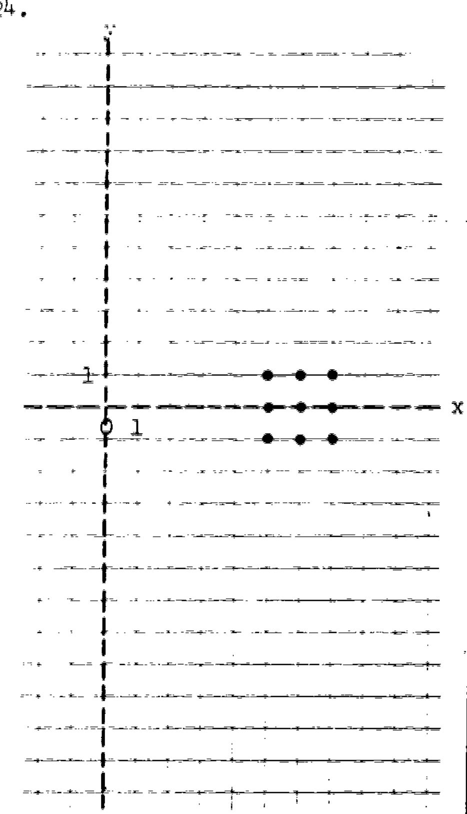
Hence, the correct choice is [A].

Write open sentences whose truth sets are the following set of points:
Turn to page xxxviii to check your work.

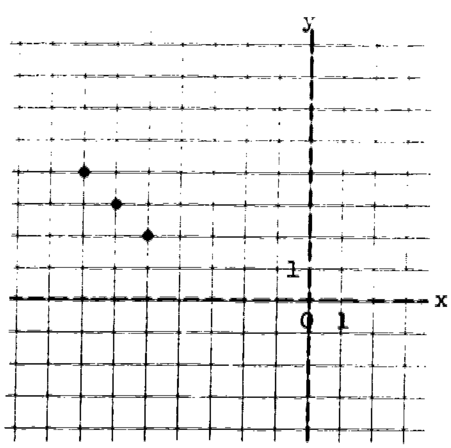
23.



24.



25.



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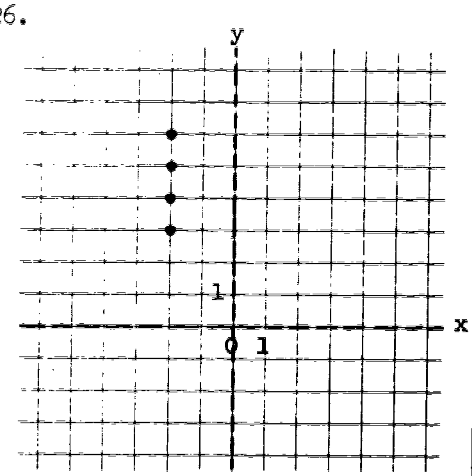
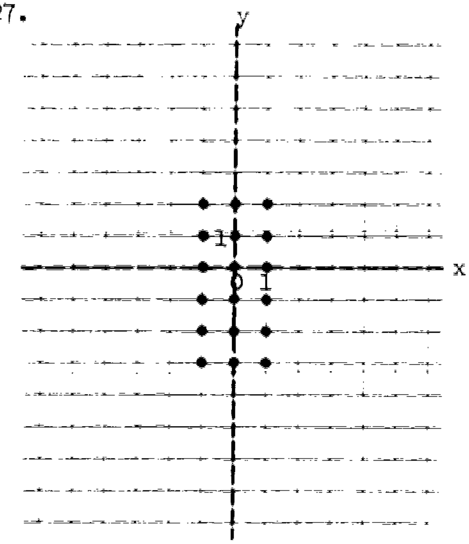
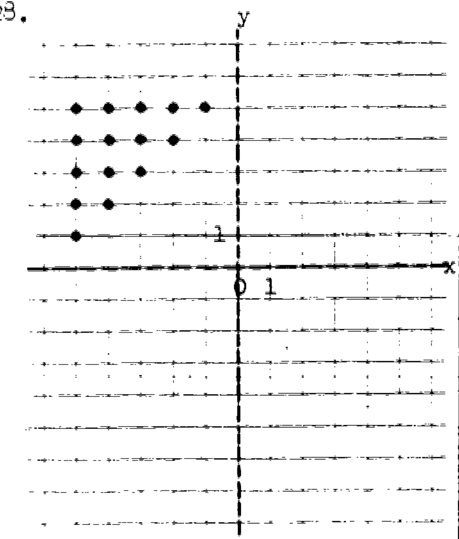


(continued)

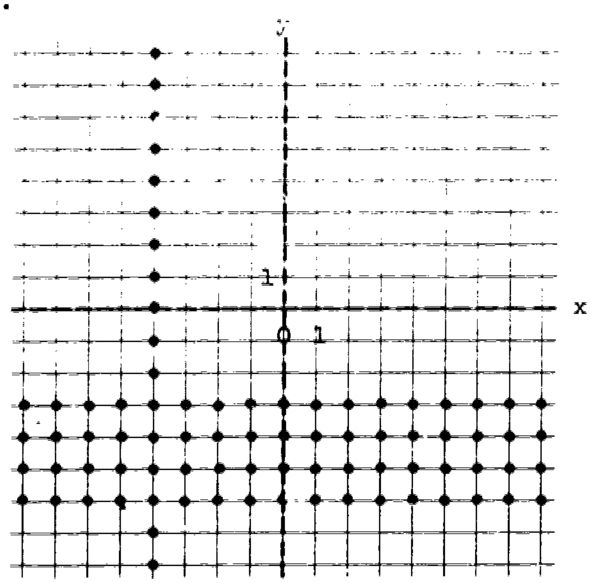
27.



28.



29.

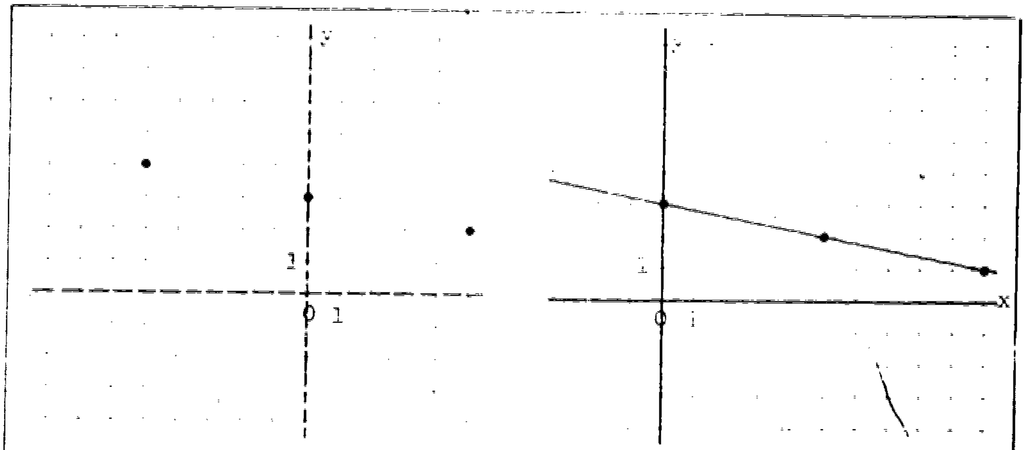


With reference to separate coordinate axes draw the graph of each of the following open sentences. Turn to pages xxxviii and xxxix to check your work.

30. $y = -2x + 3$, and $1 \leq x \leq 3$, and x and y are integers.

31. $y > \frac{1}{2}x + 1$, and $x > 0$ and $y < 4$, and x and y are integers.

32. $-3 \leq x \leq -1$ or $0 < y < 2$ and x and y are integers.



(a)

[illegible]

[illegible] ____.

and its y-intercept is ____.

The equation of (b) is y ____ [illegible]
are real [illegible].

The open sentence [illegible]
is "y ____ [illegible] x and y are ____."

(-[illegible]), ([illegible]), ([illegible],3), ([illegible]) [illegible]
graph(s)? ____
(a),(b),both

(-2,[illegible]), (1,[illegible]), ([illegible]) are [illegible]
38 graph? ____

The variables in graph (a) are restricted to ____.

40 The variables in graph (b) may be any ____ numbers.

$-\frac{1}{5}$

(0,3)

$y = -\frac{1}{5}x + 3$

$y = -\frac{1}{5}x + 3$
integers

both

(b)

integers

real

We discussed graph (a) of the open sentence $y = -\frac{1}{5}x + 3$ and x and y are integers and graph (b) of the open sentence $y = -\frac{1}{5}x + 3$ and x and y are real numbers.

If we restrict the variables to rational numbers can we draw the graph? You can guess that it would be impossible to indicate the "holes" for the exclusion of the irrational numbers.

It is hoped that this section has given you a [illegible] idea of the relation between the truth set of open sentences in a given domain and the corresponding graphs.

35[illegible]

21-4. Summary and Review

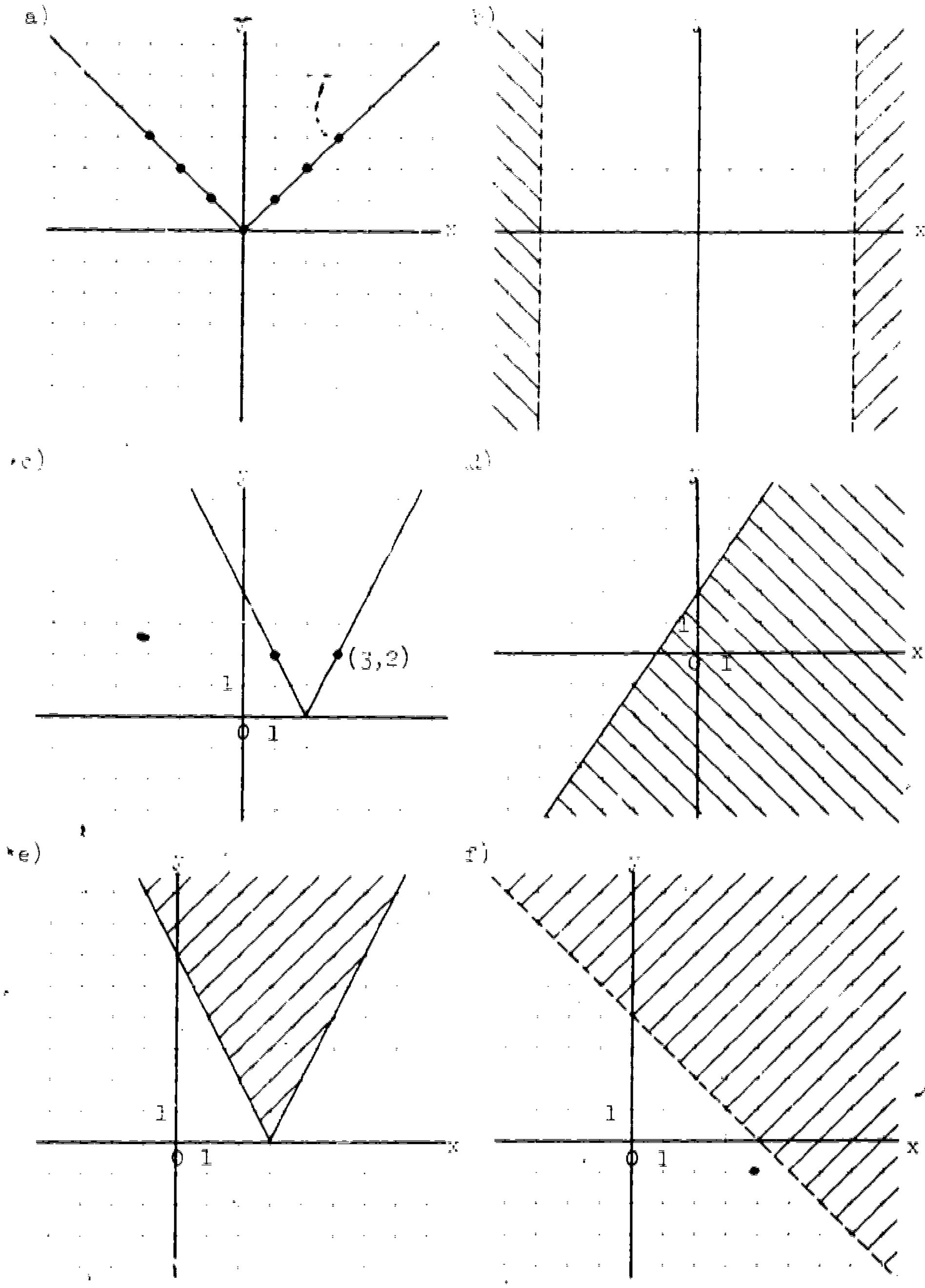
In this chapter we have looked at some graphs of open sentences which have graphs that are not lines. These may be summarized as follows:

Sentence	Graph
$y > mx + b$	The region of the plane above the line $y = mx + b$
$y \geq mx + b$	The region of the plane above the line $y = mx + b$ <u>and</u> the line $y = mx + b$.
$y < mx - b$	The region of the plane below the line $y = mx + b$.
$y \leq mx + b$	The region of the plane below the line $y = mx + b$ <u>and</u> the line $y = mx + b$.
$\lvert x \rvert = k, \quad k \neq 0$	The pair of vertical lines $x = k$ and $x = -k$.
$\lvert x \rvert = 0$	The y-axis.
$\lvert y \rvert = b, \quad b \neq 0$	The pair of horizontal lines $y = b$ and $y = -b$.
$\lvert y \rvert = 0$	The x-axis.
$\lvert x - k \rvert = b$	The pair of vertical lines a distance of k units from the line $x = b$.
$y = \lvert x \rvert$	The pair of rays intersecting at $(0,0)$ and forming a right angle.

Review Problems

Answers for the review problems are on pages xxxix - xliii.

1. Write the open sentence for each of the following graphs:



2. Draw the graph of each of the following open sentences.

(a) $y \leq 3x$

(b) $y < \frac{x}{2} -$ [illegible]

(c) $y >$ [illegible]

(d) $x <$ [illegible]

(e) $|x| + |y| = -1$

(f) $3y \geq 2x - 1$

(g) $x = 3$ and $y = -1$

(h) $x + y \leq -2$

3. Draw the graph of "$y = 2|x|$". Give an equation of the graph which results from each of the following changes.

(a) The graph is revolved one-half revolution about the x-axis.

(b) The graph is moved 3 units to the right.

(c) The graph is moved 2 units to the left.

(d) The graph is moved 5 units up.

(e) The graph is moved 2 units to the right and 4 units down.

4. Consider the sentence "$2x - 3 > 0$" and draw its graph if it is considered as an equation in

(a) one variable.

(b) two variables.

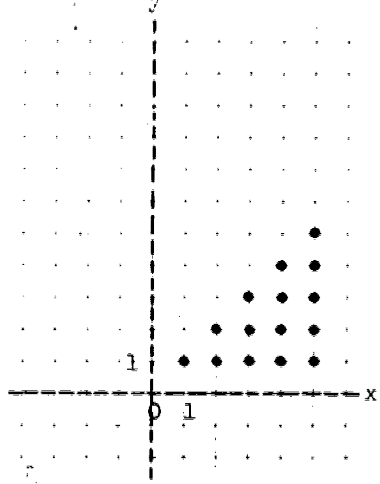
5. Consider the sentence "$|y| < 3$" and draw its graph if it is considered as a sentence in

(a) one variable.

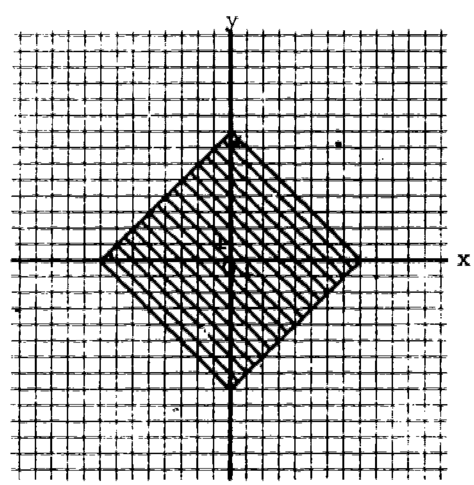
(b) two variables.

*6. Write the equations of the following:

(a)



(b)



7. If a positive integer of the form $10t + u$ is divided by the sum of its digits, the quotient is 4 and the remainder is 3. Find the possible numbers.

Remember that the sum of the digits would be $t + u$. The open sentence could be expressed as

$$\frac{10t + u}{t + u} = 4 + \frac{3}{t + u}.$$

*8. A farmer has $1000 to buy steers at $25 and cows at $26. If you know that the number of steers and the number of cows are positive integers, what is the greatest number of animals he may buy, if he must use the entire $1000?

[illegible]

[illegible] Equation

[illegible]

[illegible]

[illegible] sentence using the connective "[illegible]" [illegible]

[illegible]

[illegible]

[illegible]

However, (B) is the correct choice, since either one being true would make the compound sentence true.

[illegible] "$(x + 2y - 5)(2x + y - 1) = 0$".

[illegible]

"[illegible] = 0 or ______". — $2x + y - 1 = 0$

The truth set of this compound open sentence is the ______ (union, intersection) of the truth sets of — union

______ = 0 and $2x + y - 1$ [illegible]. — $x + 2y - 5 = 0$

In dealing with compound open sentences [illegible], it is convenient [illegible] truth sets in terms of the [illegible]. Thus, the graph of

"$x + 2y - 5 = 0$ or $2x + y - 1 = 0$"

[illegible] of the graph of $x + 2y - 5 = 0$ and the graph of $2x + y - 1 = 0$. The graph of $2x + y - 1 = 0$ is a line, [illegible] the graph of $2x + y - 1 = 0$. Thus, the graph of the compound sentence consists of [illegible] lines.

Change each of the following into an equivalent compound open sentence and draw the corresponding graph. The answers for Items 5-7 will be found on page [illegible].

5. $(x + 2y - 5)(2x + y - 5) = 0$
6. $(x + y - 2)(x - y + 4) = 0$
7. $(x - 1)(y - 3) = 0$

8	Do you notice something about these three graphs? All six of the lines which you have drawn have one point in common; namely, the point (,).	(-1,3)

Since $(x + 1)(y - 3) = 0$ is equivalent to $x + 1 = 0$ or $y - 3 = 0$, the graph of $(x + 1)(y - 3) = 0$ consists of all the points which lie on either the line $x + 1 = 0$ or on the line $y - 3 = 0$.

Such examples are of compound open sentences in two variables in which the connective is "or". Let us turn our attention to compound open sentences in two variables using the connective "and".

	A compound sentence using the connective "and" is true if and only if both clauses are true.	
	Consider "$x + 1 = 0$ and $y - 3 = 0$".	
	The truth set of $x + 1 = 0$ consists of all ordered ______ in which the first number is -1.	pairs
10	The truth set of $y - 3 = 0$ consists of all ordered pairs in which the ______ number is 3.	second
11	Thus, the truth set of "$x + 1 = 0$ and $y - 3 = 0$" has a single member, namely, the ordered pair (,).	(-1,3)
12	The truth set or solution ______ of "$x + 1 = 0$ and	set
13	$y - 3 = 0$" is ______.	{(-1,3)}

Let us adopt the form

$$\begin{cases} x + 1 = 0 \\ y - 3 = 0 \end{cases}$$

in place of "$x + 1 = 0$ and $y - 3 = 0$".

We shall speak of this as a <u>system of equations</u> in two variables. By the <u>truth set of a system of equations</u>, we mean the set of ordered pairs common to the truth sets of the individual equations; that is, the truth set of the system is the intersection of the truth sets of the equations.

As we have seen, the truth set of

$$\begin{cases} x + 1 = 0 \\ y - 3 = 0 \end{cases}$$

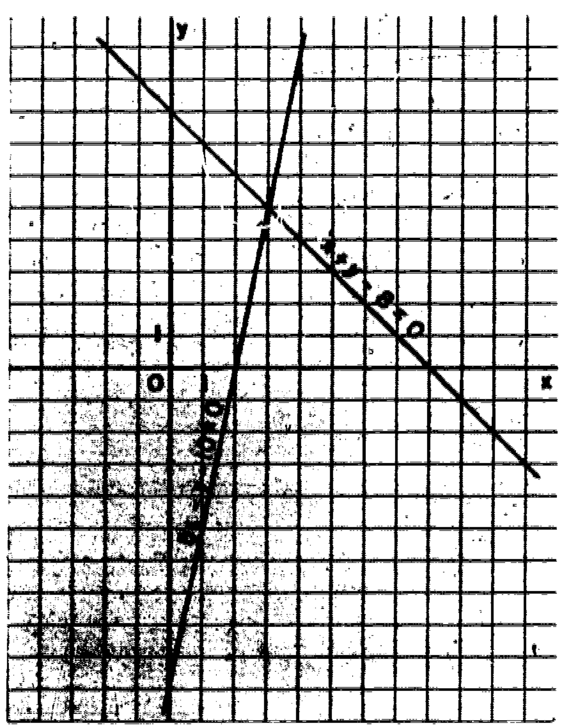
is $\{(-1,3)\}$; it is the intersection of the truth sets of $x + 1 = 0$ and of $y - 3 = 0$. Correspondingly, the graph of $x + 1 = 0$ considered as an equation in two variables is a vertical line: the graph of $y - 3 = 0$ is a horizontal line: the graph of $\{(-1,3)\}$ is the point of intersection of the two lines.

Likewise, the graph of the system

$$\begin{cases} 5x - y - 10 = 0 \\ x + y - 8 = 0 \end{cases}$$

is the intersection of the two lines that are graphs of $5x - y - 10 = 0$ and $x + y - 8 = 0$.

14	The coordinates of every point in the _______ of $5x - y - 10 = 0$ make this a true sentence.	graph
15	The coordinates of every point in the graph of _______ make $x + y - 8 = 0$ a true sentence.	$x + y - 8 = 0$
16	If a point (a,b) is the intersection of two lines $5x - y - 10 = 0$ and $x + y - 8 = 0$, then the _______ of this point make both sentences true.	coordinates
17	If (a,b) is a point of intersection of two lines, then _______ is the solution of the system of equations.	(a,b)
18	Graph the system $\begin{cases} 5x - y - 10 = 0 \\ x + y - 8 = 0 \end{cases}$	[See graph below]



19 The point of intersection of the lines appears to be ______. | (3,5)

We may verify that (3,5) <u>is</u> the solution of the system,

$$\begin{cases} x - y - 10 = 0 \\ x + y - 8 = 0 \end{cases} \quad \text{since} \quad \begin{cases} 3(3) - (5) - 10 = 0 \\ \underline{\quad + \quad - 8 = 0} \end{cases}$$

$3 + 5 - 8 = 0$

[illegible] of the following systems, draw the corresponding graph and find [illegible] point of intersection. Verify that the coordinates of the point of [illegible] satisfy the system. Answers for these items will be found [illegible] xlv and xlvi.

[illegible] $\begin{cases} [illegible] - 14 = 0 \\ [illegible] - 3 = 0 \end{cases}$

23. $\begin{cases} x - 4y - 15 = 0 \\ 3x + 5y - 11 = 0 \end{cases}$

[illegible] $\begin{cases} [illegible] - 27 = 0 \\ [illegible] -20 \end{cases}$

24. $\begin{cases} x + y - 30 = 0 \\ x - y + 7 = 0 \end{cases}$

[illegible] problem stated in English that leads to a system of equations.

[illegible] of ore of unknown weight are placed on the [illegible] of a set of scales; the scales are balanced [illegible] standard 20 gram weight is placed on the other pan. [illegible] pieces are placed one on each pan, a standard [illegible] weight must be added to the pan containing the [illegible] ore in order to balance the scales. What is the [illegible] each ore?

[illegible] equations for this problem is

$$\begin{cases} x + y = \underline{\qquad} \\ x = y + 10 \end{cases},$$

x + y = 20

[illegible] equivalently,

$$\begin{cases} x + y = 20 \\ x - y = \underline{\qquad} \end{cases}.$$

x - y = 10

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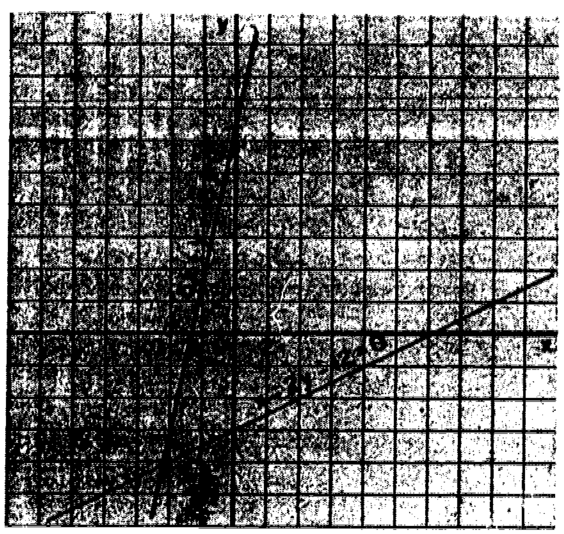
27 Draw the corresponding graph of the system.

The weights of the pieces of ore are

28 _______ grams and _______ grams.

29 Graph the system $\begin{cases} 5x - y + 13 = 0 \\ x - 2y - 12 = 0. \end{cases}$



The point of intersection of the lines appears to

30 be ______. — Any point close to $(-4.2,-8.1)$ may be given as an answer.

However, $(-4.2,-8.1)$ is not the solution of the

31 system, since $5(-4.2)-(-8.1) + 13 =$ ______ $\neq 0$. — .6

Graphing, even though carefully done, does not always yield a precise solution. However, we will see that examining the graphing procedure will be helpful in developing other methods.

Consider the following system:

$$\begin{cases} x - 2 = 0 \\ y + 5 = 0. \end{cases}$$

The truth set of this system is "obvious"; the truth

32 set is ______. — $\{(2,-5)\}$

In fact, the truth set of

$$\begin{cases} x - a = 0 \\ y - b = 0 \end{cases}$$

33 is "obvious". The truth set is ______. — $\{(a,b)\}$

Suppose we start with a system of equations whose solution set is not obvious.

For example, at the beginning of this chapter we considered the system

$$\begin{cases} x + 2y - 5 = 0 \\ 2x + y - 1 = 0. \end{cases}$$

We know (see Items 5 and 8) that this system has the

34 solution set ______. — $\{(-1,3)\}$

That is, the system $\begin{cases} x + 2y - 5 = 0 \\ 2x + y - 1 = 0 \end{cases}$

35 and the system $\begin{cases} x + 1 = 0 \\ \underline{\qquad} - 3 = 0 \end{cases}$ — $y - 3 = 0$

have the same truth set.

In other words, the two systems are equivalent.

If we wish to find the solution set of a system such as

$$\begin{cases} x + 2y - 5 = 0 \\ 2x + y - 1 = 0 \end{cases},$$

it would be convenient to find first an equivalent system; in this case, the system

$$\begin{cases} x + 1 = 0 \\ y - 3 = 0 \end{cases},$$

having the same (but now, obvious) truth set.

Our present task, then, is to develop a method of finding equivalent systems of equations. Our task is aided by examination of the graph of the equations.

Suppose we start with a pair of lines intersecting in the single point (a,b). The corresponding system of equations has the truth set $\{(a,b)\}$. Any pair of distinct lines through (a,b) corresponds to a system of equations which is equivalent to the original system. In particular, the horizontal and vertical lines through (a,b) correspond to the system

$$\begin{cases} x - a = 0 \\ y - b = 0 \end{cases}.$$

The truth set of <u>this</u> system is obvious.

We begin our discussion by finding a method of writing the equation of <u>any</u> line through the point of intersection of two given lines.

	In looking for equivalent <u>systems</u> of equations, let us recall first some ways of writing an equation equivalent to an individual equation of the system.	
36	For example, $2(x + 2y - 5) = 0$ and $\underline{x + 2y \qquad = 0}$ are equivalent.	$x + 2y - 5 = 0$
37	Similarly, $3(2x + y - 1) = 0$ is equivalent to $\underline{2x + \qquad = 0}$.	$2x + y - 1 = 0$
38	If the coordinates of a point (a,b) satisfy the equation $x + 2y - 5 = 0$, then they also satisfy the equation $\underline{2(x + \qquad) = 0}$; that is,	$2(x + 2y - 5) = 0$
39	$\underline{2(a + \qquad) = 0}$.	$2(a + 2b - 5) = 0$

Further, if the point (a,b) lies on the line $2x + y - 1 = 0$, then it satisfies the equation $3(2x + y - 1) = 0$ and

40 $$3(2a + \_\_\_\_) = 0.$$ $3(2a+b-1) = 0$

Hence, if (a,b) is a point of intersection of the

41 lines, then $a + \_\_\_\_ = 0$, and $2a + b - 1 = 0$. $a + 2b - 5 = 0$

42 So $2(a + 2b - 5) + 3(2a + b - 1) = 2 \cdot \square + 3 \cdot \square$ $2 \cdot 0 + 3 \cdot 0$

43 $= \_\_\_\_.$ 0

We see then that if (a,b) satisfies both the equation $x + 2y - 5 = 0$ and the equation $2x + y - 1 = 0$, then it also satisfies the equation

$$2(x + 2y - 5) + 3(2x + y - 1) = 0.$$

We have seen that the lines $x + 2y - 5 = 0$ and $2x + y - 1 = 0$ intersect at the point $(-1,3)$.

44 Hence, ____ satisfies the equation $(-1,3)$

$$2(x + 2y - 5) + 3(2x + y - 1) = 0.$$

The equation $2(x + 2y - 5) + 3(2x + y - 1) = 0$ may be written in the form $Ax + By + C = 0$ by multiplying and collecting terms. The equation written in this simplified form is

45 $$8x + \_\_\_\_ = 0;$$ $8x + 7y - 13 = 0$

46 the graph of this ____ is a line. equation

We can verify that the point $(-1,3)$ lies on this line; namely, that

47 $$8(-1) + \_\_\_\_ = 0.$$ $8(-1)+7(3)-13=0$

From the equations of two given lines we have found an

48 equation of another line through the point of ____ of the given lines. intersection

Let's experiment some more with the original two equations.

49 We have noticed that the equations $2(x + \_\_\_\_) = 0$ and $x + 2y - 5 = 0$ are equivalent. We can see that the $2(x+2y-5)=0$

50 equations $-(2x + y - 1) = 0$ and $2x + \_\_\_\_ = 0$ are are equivalent. $2x + y - 1 = 0$

Since (-1,3) satisfies both equations $x + 2y - 5 = 0$
51 and $2x + y - 1 = 0$, (,) satisfies **(-1,3)**

$$2(x + 2y - 5) - (2x + y - 1) = 0.$$

Another form of this equation is

52 $\underline{3y \quad\quad = 0}.$ **3y - 9 = 0**

53 The simplified form of this last equation is $\underline{y \quad\quad = 0}.$ **y - 3 = 0**

We can verify that the coordinates of the point of
54 intersection (,) satisfy this equation: $(3) - 3 = 0$. **(-1,3)**

Let's try

$$5(x + 2y - 5) + 7(2x + y - 1) = 0.$$

By the same line of reasoning, we should be able to see that (-1,3) satisfies this equation.

55 The simplified form of this equation is ______. **19x + 17y - 32 = 0**

56 Verify that (-1,3) does satisfy this equation. _____ **19(-1) + 17(3) - 32 = 0**

In the next section we shall investigate further such combinations of equations and see how to find combinations leading to equations whose truth sets are obvious.

In the preceding section, we have tried various combinations of equations of two given lines which intersect in exactly one point. Each time, we arrived at an equation of a line that passes through the point of intersection.

In particular, we worked with the system

$$\begin{cases} x + 2y - 5 = 0 \\ 2x + y - 1 = 0. \end{cases}$$

The point of intersection of the two lines of this system is $(-1,3)$.

1	We can see the equations $r(x + 2y - 5) = 0$ and $x + 2y - 5 = 0$ are equivalent if $\underline{r \neq \quad}$.	$r \neq 0$
2	This means that if (a,b) satisfies the equation $x + 2y - 5 = 0$, then it also satisfies the equation $\underline{r(x + \qquad) = 0}$.	$r(x+2y-5) = 0$
3	Similarly, the equations $s(2x + y - 1) = 0$ and $2x + y - 1 = 0$ are ______ if $s \neq 0$.	equivalent
4	If (a,b) satisfies both $x + 2y - 5 = 0$ and $2x + y - 1 = 0$, then $a + 2b - 5 = 0$ and $\underline{2a + \qquad = 0}$.	$2a + b - 1 =$
5	Hence, $r(a + \qquad) + s(2a + b - 1) = 0$.	$r(a + 2b - 5)$
6	We have seen that the lines $x + 2y - 5 = 0$ and $2x + y - 1 = 0$ intersect at the point $(-1,3)$. So ______ satisfies both equations.	$(-1,3)$
7	This means that $(-1,3)$ also ______ the equation $r(x + 2y - 5) + s(2x + y - 1) = 0$, since	satisfies
8	$r\left(-1 + 2(3) - 5\right) + s\underline{\left(2(\ ) + \quad\right)} = r \cdot 0 + s \cdot \underline{\square}$	$2(-1)+3-1$, 0
9	If r and s are non-zero real numbers, then $\underline{(\ ,\ )}$ satisfies $r(x + 2y - 5) + s(2x + y - 1) = 0$.	$(-1,3)$

We started with the system

$$\begin{cases} x + 2y - 5 = 0 \\ 2x + y - 1 = 0, \end{cases}$$

and we noted that $r(x + 2y - 5) = 0$ and $x + 2y - 5 = 0$ are equivalent; similarly, that $s(2x + y - 1) = 0$ and $2x + y - 1 = 0$ are equivalent. We do not mean to imply that the combination $r(x + 2y - 5) + s(2x + y - 1) = 0$ is equivalent to each of the equations in the original system for all real numbers r, s. The main point we want to make is that if (a,b) satisfies each equation in the system, then it satisfies the combined equation.

	In the previous section we have actually tried various values of r and s. For example, if r is 2 and s is 3, we get	
10	$2(\_\_\_\_\_\_) + 3(2x + y - 1) = 0.$	$x + 2y - 5$
	This equation can be simplified as	
11	$\underline{8x + \qquad = 0},$	$8x + 7y - 13 = 0$
12	and we verified that the point of ______ of the original two lines lies on this line.	**intersection**
	If r is 2 and s is -1, the equation $r(x + 2y - 5) + s(2x + y - 1) = 0$	
13	in simplified form is ______.	$y - 3 = 0$
14	We can also verify that (-1,3) ______ this equation.	**satisfies**
15	If r is 1 and s is -2, the equation is ______.	$x + 1 = 0$
16	(,) satisfies this equation since $-1 + 1 = 0$.	(-1,3)
17	If r is 0 and s is 1, the equation is ______.	$2x + y - 1 = 0$
18	Does (-1,3) satisfy this equation? ______ (yes,no)	**yes**
19	If r is 1 and s is 0, the equation is ______.	$x + 2y - 5 = 0$
20	(-1,3) ______ this equation.	**satisfies**
	We can see that even if one of the numbers r, s is	
21	0, the equation is that of a ______ through (-1,3).	**line**
	Suppose both $r = 0$ and $s = 0$. The coordinates	
22	(,) still satisfy the equation, but the equation	(-1,3)
23	is ___ = ___, and is not that of a line.	$0 = 0$

The various equations we obtained are as follows:

[11] $8x + 7y - 13 = 0$
[13] $y - 3 = 0$
[15] $x + 1 = 0$
[17] $2x + y - 1 = 0$
[19] $x + 2y - 5 = 0.$

	The graphs of these have a point in common; any two of	
24	these graphs have the same point of ______.	intersection
	Any two of these equations form a system of equations that is equivalent to the original system, $$\begin{cases} x + 2y - 5 = 0 \\ 2x + y - 1 = 0. \end{cases}$$	
	A particular pair of these equations is especially interesting because the solution of the system is obvious for this pair. Which two equations make up	
25	this pair? [ ] and [ ].	[See answer below]
	Any pair is equivalent; [13] and [15] are the natural choices as the pair in which the solution is obvious.	

We indicated that the system

$$\begin{cases} y - 3 = 0 \\ x + 1 = 0 \end{cases}$$

is of special interest because the solution of this system is obvious. Referring to Items 13 and 15 for the choices for r, s, which yielded these simple equations should prove profitable in that it may indicate how we may best choose particular pairs of numbers r, s, most useful to use.

	The equation $$r(x + 2y - 5) + s(2x + y - 1) = 0$$ may be rewritten	
26	$$(r + 2s)x + (\_\_\_\_)y - 5r - s = 0.$$	$2r + s$
	In Item 13 we used $r = 2$ and $s = -1$.	
27	For this choice, the coefficient of x is ______ since $\big(2 + 2(-1)\big) = 0$.	0
	The resulting equation is $y - 3 = 0$, which we think of as an equation in two variables in which the	
28	variable ______ does not appear.	x
	In Item 15, we used $r = 1$ and $s = -2$. For this	
29	choice, the coefficient of ______ is 0.	y

The resulting equation is thought of as an equation in two variables in which the variable y does not appear.

30 That is, the ______ of y is 0. **coefficient**

This gives us a hint as to how we may choose numbers r, s, in the combination, $r(x + 2y - 5) + s(2x + y - 1) = 0$. Writing the equation in the form

$$(r + 2s)x + (2r + s)y - 5r - s = 0,$$

if the coefficient of x is to be zero, then

31 $r + \_\_\_\_ = 0$ $r + 2s = 0$

32 This indicates that a choice such that $r = \_\_\_\_$ would give a very simple equation. $r = -2s$

In the same way, we can choose r, s, so that the coefficient of y is 0.

33 For this, we must have $2r + \_\_\_\_ = 0$, or $2r = -s$. $2r + s = 0$

We are led to conclude that if $Ax + By + C = 0$ and $Dx + Ey + F = 0$ meet in a point (a,b), then (a,b) is a point of the line

$$r(Ax + By + C) + s(Dx + Ey + F) = 0,$$

where r and s are real numbers, not both 0.

We can see that this is so because (a,b) satisfies both equations; that is,

$$Aa + Bb + C = 0 \quad \text{and} \quad Da + Eb + F = 0.$$

Thus, $r(Aa + Bb + C) + s(Da + Eb + F) = r \cdot 0 + s \cdot 0 = 0$, and (a,b) satisfies

$$r(Ax + By + C) + s(Dx + Ey + F) = 0.$$

This last equation can be rewritten as

$$(rA + sD)x + (rB + sE)y + rC + sF = 0.$$

* For each of the following systems, write the equation of the line through the point of intersection in the form

$$(rA + sD)x + (rB + sE)y + rC + sF = 0.$$

Answers for these items will be found on page xlvi.

Example: $\begin{cases} x + 2y - 5 = 0 \\ 2x + y - 1 = 0 \end{cases}$

$r(x + 2y - 5) + s(2x + y - 1) = 0$
$(r + 2s)x + (2r + s)y - 5r - s = 0.$

*34. $\begin{cases} x + y - 2 = 0 \\ x - y + 4 = 0 \end{cases}$

*35. $\begin{cases} 5x - y - 10 = 0 \\ x + y - 8 = 0 \end{cases}$

*36. $\begin{cases} 3x - 2y - 14 = 0 \\ 2x + 3y + 8 = 0 \end{cases}$

*37. $\begin{cases} 8x + 12y - 5 = 0 \\ -6x + 15y + 4 = 0 \end{cases}$

The equation for a line through the common point of

$$\begin{cases} x + y - 2 = 0 \\ x - y + 4 = 0 \end{cases}$$

is $(r + s)x + (r - s)y - 2r + 4s = 0$.

What is the resulting equation if $s = r$?

Replacing s by r,

*38 $(r + r)x + (\_\_\_\_)y - 2r + 4r = 0$ — $r - r$

*39 $\underline{2rx + \qquad = 0}$ — $2rx + 2r = 0$

Since $r = s$ and not both r, s, are 0, then

*40 $r \neq 0$ and _____; so we can divide by $2r$ to get — $s \neq 0$

$2rx + 2r = 0$ in the simplified form

*41 $\underline{\qquad\quad} = 0.$ — $x + 1 = 0$

Thus, in this case, if we choose $r = s$, we get

*42 a ____________ (horizontal, vertical) line through the point — vertical

*43 of _____ of the original lines. — intersection

If $s = -r$, the resulting equation is

*44 $(\_\_\_\_)x + (r + r)y - 2r - 4r = 0.$ — $r - r$

*45 Simplifying, $\underline{y \qquad} = 0$, $r \neq 0$. — $y - 3 = 0$

If $Ax + By + C = 0$ and $Dx + Ey + F = 0$ are equations of lines, by rewriting $r(Ax+By+C)+s(Dx+Ey+F) = 0$ as $(rA+sD)x + (rB+sE)y + rC+sF = 0$, we are led to some desirable choices for r, s. By choosing r and s (not both 0) so that $rB + sE = 0$, we obtain an equation of a vertical line through the point of intersection of $Ax + By + C = 0$ and $Dx + Ey + F = 0$. A different choice will make $rA + sD = 0$ and result in an equation of a horizontal line through the point of intersection of the original lines. The equations of the vertical and horizontal lines form a system whose truth set is obvious, and which is the same as the truth set of the original system.

In practice, we often do not need to rewrite the equation in the form $(rA + sD)x + (rB + sE)y + rC + sF = 0$. We can usually determine what numbers r, s to choose by inspecting the original system.

For example, for the system

$$\begin{cases} 2x - y - 2 = 0 \\ 3x + y - 3 = 0, \end{cases}$$

if we choose $r = 1$, $s = 1$, (thus, $r = s$), then we
46 can see that the coefficient of _______ in the combination will be 0. y

47 If we choose $r = 5$, $s =$ _______, the same will be true.

If we choose $r = 3$ and $s = -2$, the coefficient of
48 _______ in the combination will be 0. x

Using the choices indicated in Items 46 and 48, we find that the original system is equivalent to the system

49 $\begin{cases} x\text{\_\_\_\_\_\_} \\ \text{\_\_\_\_\_\_} \end{cases}$ $x - 1 = 0$
50 $y = 0$

51 The solution of this system is obviously _______. (1,0)

Using $r = 1$, $s = -5$, and then using another pair of numbers r, s, the solution of

$$\begin{cases} 5x - y - 10 = 0 \\ x + y - 8 = 0 \end{cases}$$

52 can be found to be (_ , _). (3,5)

The solution of

$$\begin{cases} 3x - 2y - 27 = 0 \\ 2x - 7y = -50 \end{cases}$$

53 is _______. (17,12)

The solution of

$$\begin{cases} 8x + 12y - 5 = 0 \\ -6x + 15y + 4 = 0 \end{cases}$$

54 is _______. $\left(\frac{9}{64}, -\frac{1}{96}\right)$

What did you use for r and s to make the coefficient of x equal to 0 ? Perhaps you used $r = 6$ and $s = 8$. You may have noticed that $r = 3$ and $s = 4$ is as good a choice.

55	Notice that ______ is the least common multiple of 8 and 6.	24
	So we can pick r, s, such that	
56	$8r = 24$ and $6s =$ ______;	24
57	that is, $r =$ ______ and $s =$ ______.	$r = 3$ and $s = 4$
	Let's use this idea to find an r and an s that would make the coefficient of y equal to 0. Instead of using $r = 15$ and $s = -12$, we notice that	
58	the ______ ______ ______ of 15 and 12 is 60.	least common multiple
	Therefore, we choose r, s, so that $12r = 60$	
59	and $-15s =$ ______;	$-15s = 60$
60	that is, $r =$ ______ and $s =$ ______.	$r = 5$ and $s = -4$

*61	Find the solution of $\begin{cases} 4x + 21y - 27 = 0 \\ -6x + 15y = 37 \end{cases}$ ______	$(-2, \frac{5}{3})$
*62	For Item *61 you may have found a different r and a different s than $r = 6$, $s = 4$ by examining the least common multiple. You may also have noticed that you can find an r and an s by examining greatest common factors. Try to describe how this can be done for the case $\begin{cases} 4x + 21y - 27 = 0 \\ -6x + 15y - 37 = 0. \end{cases}$ Compare your description with the one on page xlvi.	

In Items 46 to 51 we worked with the system

$$\begin{cases} 2x - y - 2 = 0 \\ 3x + y - 3 = 0 . \end{cases}$$

Using $r = 1$, $s = 1$, we wrote the equation

$$(2x - y - 2) + (3x + y - 3) = 0$$

which, when simplified, becomes

$$-5y = 0 \quad \text{or} \quad y = 0.$$

Let us examine the systems

$$\begin{cases} 2x - y - 2 = 0 \\ 3x + y - 3 = 0 \end{cases} \qquad \begin{cases} x - 1 = 0 \\ y = 0 \end{cases}.$$

If an ordered pair is an element of the solution set of the first system, then it is also an element of the solution set of the second system.

The solution set of the system

$$\begin{cases} x - 1 = 0 \\ y = 0 \end{cases}$$

63 is ______. $\{(1,0)\}$

We can verify that $(1,0)$ satisfies the original system. Since every element of the solution set of the first system is an element of the solution set of the second, we see that $(1,0)$ is the <u>only</u> solution of the first system.

Since the two systems have the same solution set; 64 namely, $\{(1,0)\}$, the systems are ______. equivalent

Examine carefully the steps in solving

$$\begin{cases} 5x - y - 10 = 0 \\ x + y - 8 = 0 . \end{cases}$$

We might use $r = 1$, $s = 1$. To indicate this, we may write simply

$5x - y - 10 = 0$ (from the first equation)
$x + y - 8 = 0$ (from the second equation).

Then $(5x - y - 10) + (x + y - 8) = 0$ is easily seen 65 to be $\underline{6x \quad\quad}$. $6x - 18 = 0$.

Similarly, we might use $r = 1$, $s = -5$. To indicate this, we may write:

$$\begin{cases} 5x - y - 10 = 0 \\ -5x - 5y + 40 = 0. \end{cases}$$

Then $(5x - y - 10) - 5(x + y - 8)$ is seen to 66 be $\underline{-6y \quad\quad}$. $-6y + 30 = 0$

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Simplifying the equations in Items [illegible] and [illegible], we obtain the system

$$\begin{cases} x - \underline{\quad} = 0 \\ y - \underline{\quad} = 0 \end{cases}$$ $x - 3 = 0$; $y - 5 = 0$

69 This system has the solution set $\{\underline{\quad\quad}\}$. $\{(3,5)\}$

We can verify that $(3,5)$ is a solution of the original system.

Thus, we can conclude: the systems

$$\begin{cases} 5x - y - 10 = 0 \\ x + y - 8 = 0 \end{cases} \qquad \begin{cases} x - 3 = 0 \\ y - 5 = 0 \end{cases}$$

70 are ______. equivalent

In solving systems of equations, you may prefer to write your steps in the vertical form shown above. The method of solving systems of equations we have shown is sometimes called the <u>addition</u> method.

71 Solve $\begin{cases} 5x - y + 13 = 0 \\ x - 2y - 12 = 0 \end{cases}$ Solution set $\{(\quad,\quad)\}$ $\{(-4\frac{2}{9}, -8\frac{1}{9})\}$

72 Find the solution of $\begin{cases} 3x - 2y - 14 = 0 \\ 2x + 3y + 8 = 0 \end{cases}$ ______. $(2,-4)$

73 Find the solution of $\begin{cases} 5x + 2y = 4 \\ 3x - 2y = 12 \end{cases}$ ______. $(2,-3)$

74 The lines $3x - 2y = 27$ and $2x - 7y = -50$ intersect in the point ______. $(17,12)$

75 Solve $\begin{cases} x + y - 30 = 0 \\ x - y + 7 = 0 \end{cases}$ Truth set: $\{\underline{\quad\quad}\}$. $\{(\frac{23}{2}, \frac{37}{2})\}$

76 Solve $\begin{cases} \frac{1}{2}x + y = 2 \\ y - \frac{1}{3}x = 1 \end{cases}$ Truth set: $\{\underline{\quad\quad}\}$. $\{(\frac{6}{5}, \frac{7}{5})\}$

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Let us use the addition method to find the solution of each of the following systems of equations.

77 $\begin{cases} x - 4y - 15 = 0 \\ 3x + 5y - 11 = 0 \end{cases}$ ______ **(7,-2)**

78 $\begin{cases} 2x = 3 - 2y \\ 3y = 4 - 2x \end{cases}$ ______ **$(\frac{1}{2},1)$**

79 $\begin{cases} 2x = 3 - 2y \\ 3y = 4 - 2y \end{cases}$ ______ **$(\frac{7}{10}, \frac{4}{5})$**

80 $\begin{cases} 2x = 3 - 2y \\ 3y = 4 - 3x \end{cases}$ Truth set: ______ **$\emptyset$**

Did you discover that your work led to an impossible conclusion in working with Item 80? We will investigate this type of system in more detail in the next section.

Sometimes problems stated in English lead to systems of equations.

Tickets for a basketball game are 25 cents for students and 75 cents for adults.

81 Suppose x ______ were sold to students and y tickets **tickets**
82 were sold to ______, with a total sale of 311 tickets. **adults**

We know that 311 tickets were sold. Hence, we are
83 led to the equation $x + y =$ ______. **x + y = 311**

25x represents the receipts (in cents) from student tickets sold.

84 ______ represents the receipts (in cents) from adult tickets sold. **75y**

If we know that the total receipts were $103.75, we
85 are led to the equation ______ = 10875. **25x+75y=10875**

Solving the system $\begin{cases} x + y = 311 \\ 25x + 75y = 10875 \end{cases}$
86 we have the truth set ______. **{(249,62)}**

87 Hence, there were ______ student and ______ adult tickets sold. **249, 62**

Let's have some fun:

A family is coming to visit and we have to provide beds, but no one seems to know how many children there are in the family.

Elsie, one of the daughters, writes that she has as many brothers as sisters.

If x represents the number of daughters, then Elsie
88 has _______ sisters. — **$x - 1$ [Elsie is not her own sister!]**

Further, if y is the number of sons, we know that
89 $y =$ _______. — **$y = x - 1$**

Fortunately, Ed (one of the sons) writes that he has twice as many sisters as brothers.

90 Of course, Ed has _______ sisters and _______ brothers. — **x, $y - 1$**

91 So we know that $x =$ _______. [Hint: The number of Ed's sisters is twice the number of brothers.] — **$x = 2(y - 1)$**

We then have the system
92 $\{$ _______ $= 0$ — **$x - y - 1 = 0$**
93 $\{$ _______ $= 0$ — **$x - 2y + 2 = 0$**

94 Solving, we find that there are _______ daughters — **4**
95 and _______ sons. — **3**

Try one on your own:

A class bought some three-cent and some four-cent stamps to mail bulletins.

They spent $12.67 on a total of 352 stamps.

96 They bought _______ three-cent and _______ four-cent stamps — **141, 211**

$$\begin{cases} x + y - 352 = 0 \\ 3x + 4y - 1267 = 0 \end{cases}$$

On a bank teller's account sheet, the following information was entered for the tally of one-dollar and five-dollar bills in a particular transaction.

Total number of bills	154
Total amount collected	$465

Show that something was wrong in the tally.

	This problem leads to the system of equations:	
97	$\begin{cases} \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \end{cases}$	$x + y = 154$, or $x + y - 154 = 0$
98		$x + 5y = 465$, or $x + 5y - 465 = 0$
99	The solution of the system is ______.	$(76\frac{1}{4}, 77\frac{3}{4})$
100	This points out an error because the domain is the set of positive ______.	positive integers

We have been considering some open sentences in two variables. We have been interested in discovering the truth set of such a system and in drawing its graph. Our main method of solution has been to obtain an equivalent system whose truth set is obvious. In the following section we shall discuss some special cases of open sentences in two variables.

22-3. Parallel and Coincident Lines; Solution by Substitution

In the last section we considered, for the most part, situations in which a system of equations had a truth set consisting of a single number pair. We could equally well say that the graphs of the separate equations of each system have intersected in a single point.

Consider the two lines

$$2x + y - 4 = 0$$

and

$$2x + y - 2 = 0.$$

	Starting with $\begin{cases} 2x + y - 4 = 0 \\ 2x + y - 2 = 0 \end{cases}$	
1	we ______ both sides of the first equation by -1 to make the coefficients of y opposites.	multiply
	We form the equation $-(2x + y - 4) + (2x + y - 2) = 0$	
2	or ______ $= 0$.	$2 = 0$
3	It is reasonably clear that $2 = 0$ is a ______ (true,false) sentence.	false
	How does a false sentence arise from applying our method of solution to this system of equations?	

We can argue as follows:

4 If there is an ordered ______ of real numbers (a,b) such that $2a + b - 4 = 0$ and $2a + b - 2 = 0$ are — **ordered pair**

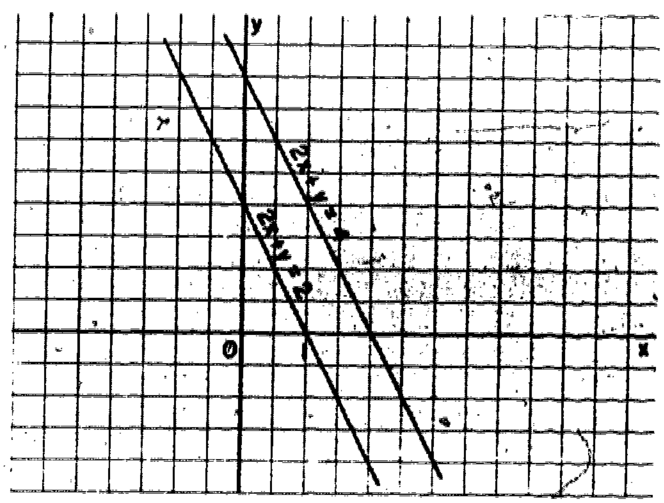
5 both true, then $2 = 0$ must also be ______. — **true**

But $2 = 0$ is false. Therefore, there is no ordered pair of real numbers which satisfies both equations of the original system of equations. We have proved that

6 the truth set of $\begin{cases} 2x + y - 4 = 0 \\ 2x + y - 2 = 0 \end{cases}$ is ______, by the indirect method of proof (proof by contradiction). — **$\emptyset$**

What would we have obtained if we had tried to graph this system of equations?

7 Graph the system $\begin{cases} 2x + y - 4 = 0 \\ 2x + y - 2 = 0 \end{cases}$. — **[See answer below]**



The graph suggests that examining the slopes of the two lines might have revealed that the lines are parallel. Recall that in Chapter 20 we have noted that if the lines are parallel, the slopes are equal and the y-intercepts are different.

8 The slope of $2x + y - 4 = 0$ is ______. — **-2**

9 The slope of $2x + y - 2 = 0$ is ______. — **-2**

10 The y-intercept of $2x + y - 4 = 0$ is ______. — **(0,4)**

11 The y-intercept of $2x + y - 2 = 0$ is ______. — **(0,2)**

12 Therefore, the two lines are ______. There is no point which lies on both lines. **parallel**

13 Hence, the truth set of $\begin{cases} 2x + y - 4 = 0 \\ 2x + y - 2 = 0 \end{cases}$ is ______. **$\emptyset$**

---

14 The truth set of $\begin{cases} 4x - 7y + 2 = 0 \\ 8x - 14y + 1 = 0 \end{cases}$ is ______. **$\emptyset$**

15 Again in this case the lines both have the same slope; namely, ______; but the y-intercepts $(0, \frac{2}{7})$ and $(0, \frac{1}{14})$ are different. **$\frac{4}{7}$**

We notice a different situation if we examine the system

$$\begin{cases} 2x - y - 5 = 0 \\ 4x - 2y - 10 = 0. \end{cases}$$

16 We multiply the first equation by ______ to make the coefficients of x opposites, then adding, the result **-2**

17 is ______ = 0. **0 = 0**

18 It is reasonably clear that $0 = 0$ is a ______ (true, false) sentence. **true**

Note that the equation $2x - y - 5 = 0$ is equivalent to the equation $4x - 2y - 10 = 0$ since we can obtain the second by multiplying both sides of the first equation by 2.

Hence, every ordered number pair which satisfies $2x - y - 5 = 0$ also satisfies $4x - 2y - 10 = 0$.

We could state this result in terms of the graphs of the corresponding lines: Every point which lies on one of the lines also lies on the other.

The graph of $\begin{cases} 2x - y - 5 = 0 \\ 4x - 2y - 10 = 0 \end{cases}$ consists of a single line.

When we try our method of solution on the system $\begin{cases} 2x - y - 5 = 0 \\ 4x - 2y - 10 = 0 \end{cases}$, we obtain $0 = 0$ which is

19 certainly ______, but this sentence does not seem to provide us with any new information. However, from the observation we just made, if we do apply our method of solution to a given system of equations and obtain **true**

20 0 = ______, we know how to interpret this. **0 = 0**

21 Every point on the line $2x - y - 5 = 0$ is also on the line $\_\_\_\_ -2y - 10 = 0$. **$4x - 2y - 10 = 0$**

22 Since there are infinitely many points on the line $2x - y - 5 = 0$, the truth set of $\begin{cases} 2x - y - 5 = 0 \\ 4x - 2y - 10 = 0 \end{cases}$ is a(n) ______ set. **infinite**

---

23 The truth set of $\begin{cases} x + y - 2 = 0 \\ 3x + 3y - 6 = 0 \end{cases}$ is the set of all ordered pairs that are ______ of points on the line. **coordinates**

24 (Be sure you see that we are dealing with only one ______.) **line**

We have seen that for the truth set of a system of equations $\begin{cases} Ax + By + C = 0 \\ Dx + Ey + F = 0 \end{cases}$ there are three possibilities:

(1) The truth set contains exactly one ordered pair;

(2) The truth set is $\emptyset$;

(3) The truth set is an infinite set of ordered pairs.

We may translate these possibilities into statements about the corresponding lines.

Given two lines whose equations are $Ax + By + C = 0$ and $Dx + Ey + F = 0$, then there are three possibilities:

25 (1) The lines intersect in exactly ______ (how many) point(s). **one**

26 (2) The lines are parallel; they do not ______. **intersect**

27 (3) The two lines are really only ______ line and, hence, every point of one lies on the other. **one**

Suppose we are given a particular system of equations. It would seem that we should be able to foresee which case we have without either graphing the lines or solving the system. Certainly we have discerned that the differences among the cases are related to the slopes of the corresponding lines.

To determine the slope of a line, we write an equation

28 of the line in y - ______. **y-form**

Thus, to find the slope of $3x + 4y - 13 = 0$ we

29 write $y =$ ______. $y = -\frac{3}{4}x + \frac{13}{4}$

30 The slope is ______. $-\frac{3}{4}$

Of course, we can also see that the y-intercept of

31 this line is ______. $(0, \frac{13}{4})$

Starting with the equations of two lines, we can write

32 both in y - ______. **y-form**

33 If the ______ are different, the lines are not parallel, nor are the lines the same line. **slopes**

If the slopes are the same, but the y-intercepts are

34 different, the lines are ______. **parallel**

35 If the slopes are the same, and the ______ are also the same, then the "two" lines are really only one line. **y-intercepts**

For each of the following systems, determine the number of elements in the truth set.

Respond "0", "1", or "infinitely many" as appropriate.

It is not necessary to graph the lines nor to solve the systems.

36 $\begin{cases} 3x + 4y - 13 = 0 \\ 5x - 2y + 13 = 0 \end{cases}$ ______ 1

37 $\begin{cases} 6x + 3y - 5 = 0 \\ 12x + 4y + 2 = 0 \end{cases}$ ______ 1

38 $\begin{cases} x - 2y - 5 = 0 \\ 3x - 6y - 12 = 0 \end{cases}$ ______ 0

39 $\begin{cases} 5x - 4y + 2 = 0 \\ 10x - 8y + 4 = 0 \end{cases}$ ______ **infinitely many**

40 $\begin{cases} 2x - y - 7 = 0 \\ 5x + 2y - 4 = 0 \end{cases}$ ______ 1

41 Which of the following lines is different from, but parallel to, the line $x = \frac{3}{2}y - 1$?

[A] $3y + 2x = 1$ [C] $\frac{y}{2} = \frac{x}{3} + \frac{1}{3}$;

[B] $3y - 2x - 1 = 0$ [D] $2x - 3y + 2 = 0$

The given line ... [illegible] [A] [illegible]
both equations [illegible]

Let us examine one of the preceding systems more carefully.

$$\begin{cases} 2x - y - 7 = 0 \\ 5x + 2y - 4 = 0 \end{cases}$$

Writing each equation in y-form, we have:

42 $\begin{cases} \_\_\_\_\_\_ \\ \_\_\_\_\_\_ \end{cases}$ $y = 2x - 7$, $y = -\frac{5}{2}x + 2$

43 We see that the two lines intersect in ______ (how many) point(s). one

---

At this point of intersection, the value of y for the equation $y = 2x - 7$ must be the same as the value of

44 y for the equation $y =$ ______. $y = -\frac{5}{2}x + 2$

45 Hence, at that point, ______ $- 7 = -\frac{5}{2}x + 2$. $2x - 7$

46 This last sentence leads to $x =$ ______, an equation whose truth set is {2}. $x = 2$

If the abscissa of a point on $y = 2x - 7$ is 2, the ordinate of this point is $y = 2(2) - 7 = -3$. Also,

47 if the ordinate of $y = -\frac{5}{2}x + 2$ is $y = -\frac{5}{2}(2) + 2 =$ ______. -3

If the abscissa is 2, the ordinates on both lines are the same since the point of intersection lies on

48 both ______. lines

We have discovered another "method" for solving a system of two equations. The method may be summarized as follows:

(a) Write each equation in y-form.

(b) Set the two expressions for y equal to each other. The resulting equation involves only the variable x.

(c) Solve this sentence, thus determining a value of x.

(d) Use this value of x to determine the value of y from one of the original equations. Use the other equation to check the work.

Using this technique, let us try to solve

$$\begin{cases} x + y = 7 \\ 2x - 3y = 4 \end{cases}.$$

49 Write $y = \underline{-x \quad\quad}$ and $y = \underline{\quad\quad}$.

Hence, we have for the point of intersection:

50 $-x + 7 = \underline{\quad\quad}$.

Solving the equation $-x + 7 = \frac{2}{3}x - \frac{4}{3}$ we have the

51 truth set ______.

If x is 5, then using either original equation we

52 find that y is ______.

53 So the solution of the system is ______.

Consider again the system $\begin{cases} x + y = 7 \\ 2x - 3y = 4 \end{cases}.$

We could shorten our work somewhat (and avoid the fractions) by writing only the first equation in y-form.

54 Thus, $\underline{y = \quad\quad}$.

55 Then we replace "y" in $2x - 3y = 4$ by "$\underline{-x + \quad\quad}$".

56 We have, then, $\underline{2x - 3(\quad\ ) = 4}$.

Solving, we obtain $2x + 3x - 21 = 4$

57 $\underline{5x = \quad\quad}$ — $5x = 25$

58 $\underline{x = \quad\quad}$ — $x = 5$

Since $y = -x + 7$, at the point of intersection, and we have $x = 5$, we can write: $y = -5 + 7$

59 $= \underline{\quad\quad}$. — 2

60 Finally, the solution of the system is ______. — (5,2)

The method just described is called the <u>substitution</u> method.

When we use the substitution method, we solve one of the
61 equations for y in terms of ______. x
62 Then substitute this expression for ______ in the other equation. y

Find the solution of the following systems by substitution.

63 $\begin{cases} 2x - y - 7 = 0 \\ 5x + 2y - 4 = 0 \end{cases}$ Solution: (,) **(2,-3)**

64 $\begin{cases} 2x - y + 13 = 0 \\ x + 4 = 0 \end{cases}$ Solution: ______ **(-4,5) [See following items]**

The solution of the system of equations in Item 64 is (-4,5). You might have noticed that if you tried to solve this system by writing the first equation in
65 y-form, we have $y =$ ______. **$y = 2x + 13$**

However, there is no y-form for the second equation,
66 ______ $= 0$. The graph of this equation $x + 4 = 0$ is a vertical line. This line is not parallel to $y = 2x + 13$; so the two lines do intersect. **$x + 4 = 0$**

But, we can carry out the same reasoning for x that we did for y.

67 That is, at the point of ______ the value of x for the equation $2x - y + 13 = 0$ must be the same as the **intersection**
68 value of x in the equation, ______. **$x + 4 = 0$**

Expressing x in terms of y in the first equation
69 $2x - y + 13 = 0$, we have $x =$ ______. **$x = \frac{1}{2}y - \frac{13}{2}$**

Finding the value of x in the second equation,
70 $x + 4 = 0$, is especially easy; it is $x =$ ______. **$x = -4$**

The last two items lead to

71 $\frac{1}{2}y -$ ___ $=$ ___ , **$\frac{1}{2}y - \frac{13}{2} = -4$**

72 from which we get $y =$ ______. **$y = 5$**

We see then that instead of solving one equation for y in terms of x, we could, of course, solve one equa-

73 tion for x in terms of ______. **y**

74 Then we would ______ this expression for x in the other equation. **substitute**

75 This would also be using the ______ method. **substitution**

To solve $\begin{cases} x - 3y + 4 = 0 \\ x + 7y - 11 = 0 \end{cases}$ we can first

76 write $\underline{x = 3y \quad\quad}$. **$x = 3y - 4$**

77 Then, "substituting", we have $\underline{(\quad) + 7y - 11 = 0}$. **$(3y-4)+7y-11=0$**

Hence, $y = \frac{3}{2}$.

Setting $y = \frac{3}{2}$ in either of the original equations

78 yields $\underline{x = \quad\quad}$. **$= \frac{1}{2}$**

79 Hence, the truth set of the system is $\{\quad\}$. **$\{(\frac{1}{2}, \frac{3}{2})\}$**

We have discussed various method or techniques for solving a system of equations. Only practice and experience will enable you to choose a "best" method for a particular system.

Find the solution of the following systems, using any convenient method.

80 $\begin{cases} 3x + y + 18 = 0 \\ 2x - 7y - 34 = 0 \end{cases}$ ______ **(-4,-6)**

81 $\begin{cases} y = \frac{2}{3}x + 2 \\ y = -\frac{5}{2}x + 40 \end{cases}$ ______ **See page xlvii.**

82 $\begin{cases} 5x + 2y - 5 = 0 \\ x - 3y - 18 = 0 \end{cases}$ ______ **(3,-5)**

83 $\begin{cases} x = \frac{3}{2}y - 4 \\ y = -\frac{2}{3}x \end{cases}$ ______ **See page xlvii.**

84 $\begin{cases} 3x + 2y = 1 \\ 2x - 3y = 18 \end{cases}$ ______ **(3,-4)**

85 $\begin{cases} x = 2y - \frac{1}{6} \\ 2x + y = \frac{4}{3} \end{cases}$ ______ $(\frac{1}{2}, \frac{1}{3})$

86 $\begin{cases} \frac{x}{2} - \frac{x}{3} = 1 \\ x + y = 7 \end{cases}$ ______ See page xlvii.

87 $\begin{cases} 5 - (x + y) = 2y \\ 2x - (3y + 1) = 1 \end{cases}$ ______ $(\frac{7}{3}, \frac{8}{9})$

88 $\begin{cases} 6y + (2 - 4x) = 3 \\ 4x - 2(3y - 1) = 2 \end{cases}$ ______ See page xlvii.

Here are some more examples of problems, stated in English, that lead to systems of equations.

Find two numbers whose sum is 56 and whose difference is 18.

89 A system of equations for this problem is { ______. $\begin{cases} x + y = 56 \\ x - y = 18 \end{cases}$

90 The numbers are ______ and ______. 37 and 19

The sum of Polly's age and Carol's age is 30 years.

Five years from now the difference in their ages will be four years. What are their ages now?

Write a system of equations for this problem. (Hint: What is the difference of their ages now?)

91 { ______. $\begin{cases} x + y = 30 \\ x - y = 4 \end{cases}$

92 Their ages are now ______ years and ______ years. 17, 13

93 Can you tell how old Polly is? ______ (yes,no) no

A dealer in nuts has cashews selling at \$1.20 a pound and almonds at \$1.50.

How many pounds of each should be mixed to obtain a 200-pound mixture to sell at \$1.32 a pound?

94 A system of equations for this problem is { ______. $\begin{cases} c + a = 200 \\ 120c + 150a = (200)(132) \end{cases}$

95 He needs ______ pounds of cashews and ______ pounds almonds. 120, 80

It takes a boat $1\frac{1}{2}$ hours to go 12 miles downstream and 6 hours to return.

Find the rate of the current c and the rate of the boat b.

96	Downstream the total rate is b + ______. (The current helps!)	b + c
97	Upstream the total rate is ______.	b - c
98	So a system of equations for this problem is $\{$ ______.	$\begin{cases}\frac{3}{2}(b+c)=12\\ 6(b-c)=12\end{cases}$
99	The rate of the current is ______ miles per hour.	3
100	The rate of the boat is ______ miles per hour.	5

A 90 % solution of alcohol is to be mixed with a 75 % solution to make 20 quarts of a 78 % solution.

How much of each should be used?

x quarts of 90 % solution will yield .9x quarts of alcohol.

101	y quarts of 75 % solution will yield ______ quarts of alcohol.	.75y
102	We need .78(20) ______ of alcohol.	quarts
103	So, we wish .9x + ____ = (.78)(20).	.9x + .75y = (.78)(20)
104	On the other hand, x + y = ______.	20
105	Hence, we need ______ quarts of 90 % solution and	4
106	______ quarts of 75 % solution.	16

Two jet planes are 400 miles apart, flying in the same direction.

One will overtake the other in 2 hours.

If they flew toward each other, they would meet in 20 minutes. How fast is each flying?

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107 Rate equals ______ divided by time.

108 The difference in their rates is $\frac{400}{2} =$ ______ mph.

109 The sum of their rates is $\frac{\square}{\frac{1}{3}} = \square$ mph.

110 The two rates are ______ mph and ______ mph.

[redacted]

Systems of equations, as you see, arise in a variety of situations. In recent years, the application of systems of inequalities have become increasingly important. Before reading Section 22-4, you might wish to think how we might investigate systems of inequalities.

22-4. Systems of Inequalities

In the preceding section, we saw examples of problems that led to systems of equations in two variables. For example, in the case of the cashew nut dealer, we obtained one equation expressing the total number of pounds of nuts in the mixture and one expressing how much the mixture was worth.

We have also seen before that quite often problems may lead to inequalities. Consider the following example.

Percy's mother sent him to the post office with a dollar for some 5-cent and 8-cent stamps. At the post office, Percy forgot how many of each kind he was supposed to get, but he did remember there were to be less than 15 stamps altogether, and he remembered his mother reminding him not to forget the change. The question is: How many of each kind of stamps does he buy?

It will turn out that the solution set for this problem contains more than one ordered pair, but that there will be a finite number of such solutions. We shall lead up to the solution in stages throughout this chapter.

The problem leads to the system $\begin{cases} x + y < 15 \\ 5x + 8y < 100. \end{cases}$

Before examining this section, let us review how we graph an inequality.

	In order to graph $x + y < 15$, we first write the sentence in y-form.	
1	Thus, $y <$ ____.	$y < -x + 15$
2	Then we graph the line $y =$ ____ and shade the	$y = -x + 15$
3	region ____ (above, below) this line.	below
	The last response is "below", since we wish to indicate all those points whose ordinate is <u>less</u> than the corresponding point on the line $y = -x + 15$.	
	Recall that the line $y = -x + 15$ is drawn with	
4	a ____, rather than with a solid line, since the	dashed
5	graph of $y < -x + 15$ ____ (does, does not) include the points of the line.	does not
	If you need further review, refer to Section 21-1.	

To graph the system (or, more accurately, the truth set of the system),

$$\begin{cases} x + y < 15 \\ 5x + 8y < 100 \end{cases}$$

we graph the truth set of each inequality separately and then identify those points that the two graphs have in common. Remember that our notation implies that we are considering the compound sentence

"$x + y < 15$ <u>and</u> $5x + 8y < 100$".

6	Graph $x + y < 15$. Shade like this ////.	[See p. xlvii.]
7	Graph $5x + 8y < 100$. Shade like this \\\\\\\\.	[See p. xlvii.]
8	Now, combine the graphs of Items 6 and 7 in the space provided for Item 8, using the proper shading.	[See p. xlvii.]

9 The graph of $\begin{cases} x + y < 15 \\ 5x + 8y < 100 \end{cases}$ consists of those parts that are

[A] shaded //// .

[B] shaded \\\\\\\\\\ .

[C] shaded XXXX.

[D] unshaded.

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The region of [A] includes the point (0,13). Does (0,13) satisfy both inequalities? Region [B] includes (16,0). Does (16,0) satisfy both inequalities? The points of region [D] satisfy neither inequality. [C] is the correct choice.

Although this doubly shaded region is the graph of the system, we have not yet given a complete answer to Percy's problem. The solution set of the problem is in this region and we shall return to this matter after we discuss the graphs of similar systems of inequalities.

A careful examination of the graphs that you have drawn in response to Item 8 shows that different regions of the plane are shaded differently. We have seen that the doubly shaded region is the graph of $\begin{cases} x + y < 15 \\ 5x + 8y < 100 \end{cases}$. Suppose these inequalities were reversed so that we are considering the system $\begin{cases} x + y > 15 \\ 5x + 8y > 100 \end{cases}$. We can, of course, graph each of these inequalities as we did for the other system; with the graph of the first system at hand, let's think about $\begin{cases} x + y > 15 \\ 5x + 8y > 100 \end{cases}$ without actually graphing.

10 The graph of $\begin{cases} x + y > 15 \\ 5x + 8y > 100 \end{cases}$ consists of all the points shown on the response to Item 8 which are:

[A] singly shaded. [B] doubly shaded. [C] unshaded.

[A] is the graph of $x + y < 15$ or $5x + 8y < 100$ but not of $x + y < 15$ and $5x + 8y < 100$. The union of this graph and the lines $x + y = 15$, $5x + 8y = 100$ make up all the points of the plane that do not belong to the graph of $x + y > 15$ and $5x + 8y > 100$.

[B] is the graph of the original system of inequalities (using "<").

[C] is the graph of those points satisfying $x + y > 15$ and $5x + 8y > 100$; hence, [C] is correct.

It should be clear that we may, following the general procedure of our illustrative example, solve other systems of inequalities, involving $>$, $\geq$, $<$, $\leq$. To graph any such system, graph the truth set of each clause separately. The truth set of the system consists of all points which belong to the graph of both clauses. Remember that it is usually convenient to write the inequalities in y-form.

Here are some practice problems. Don't forget that a solid line indicates that the boundary is included in the graph. To exclude a boundary, use a dashed line. Answers will be found on page xlvix.

Graph the following systems:

11. $\begin{cases} 6x + 3y < 0 \\ 4x - y < 6 \end{cases}$

12. $\begin{cases} 2y - 3 \geq 0 \\ -x + y + 1 \geq 0 \end{cases}$

13. $\begin{cases} 2x + y > 4 \\ 2x + y < 6 \end{cases}$

14. $\begin{cases} 2x + y < 4 \\ 2x + y > 6 \end{cases}$

15. $\begin{cases} 2x - y \leq 4 \\ 4x - 2y < 8 \end{cases}$

16. $\begin{cases} x + 2y - 4 > 0 \\ 2x - y - 3 > 0 \end{cases}$

For the sentence $x + 2y - 4 > 0$, the y-form is

$$y > -\frac{1}{2}x + 2.$$

The graph consists of all the points above the line $y = -\frac{1}{2}x + 2$. We can, instead, write the inequality in the equivalent form $x > -2y + 4$. Then the graph consists of all the points to the right of the line $x = 2y + 4$.

We can verify that the lines $y = -\frac{1}{2}x + 2$ and $x = -2y + 4$ are equivalent. So the above comment indicates that for this graph, the set of points above the line $x + 2y - 4 = 0$ and the set of points to the right of this line are identical. Verify for yourself by referring to the graph that these two sets of points are the same set.

Let's discuss now the solution for our post office problem in more detail. In this connection, we considered the system of inequalities

$$\begin{cases} x + y < 15 \\ 5x + 8y < 100 \end{cases},$$

and the graph corresponding to this system. This is the region below both dashed lines in the following graph.

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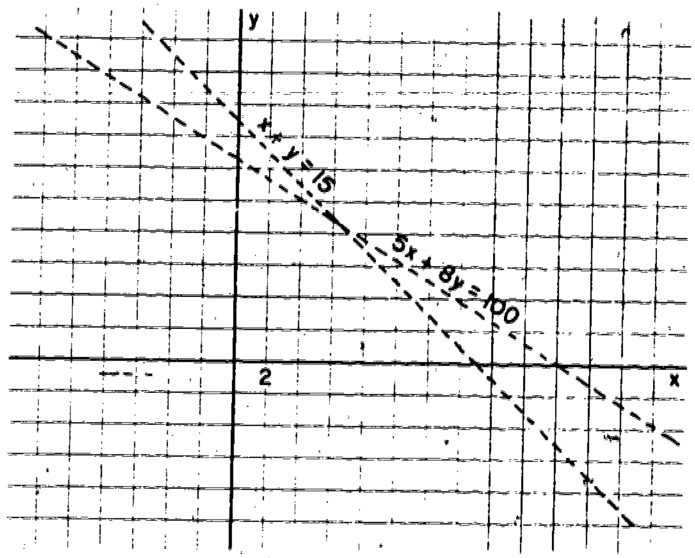
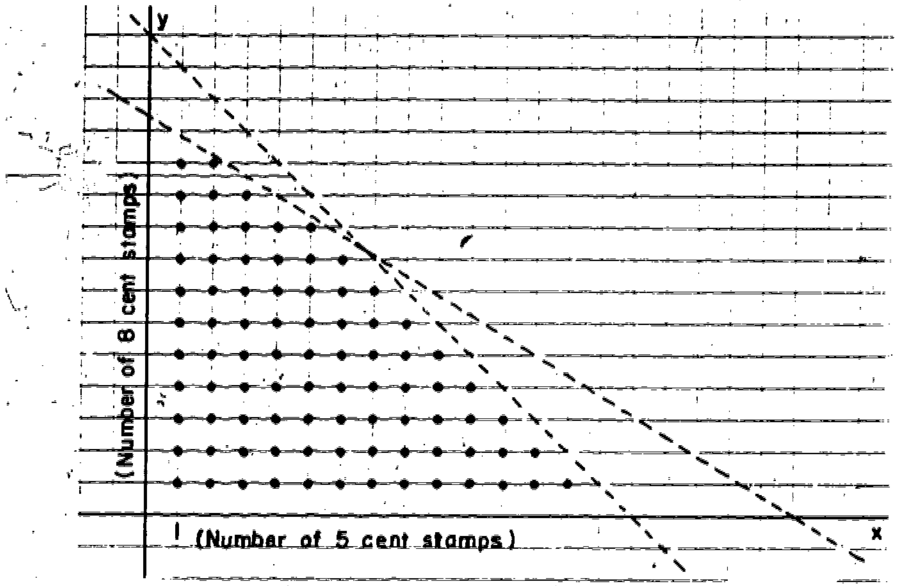


What has not been said but what we would like to mention now are some unspoken agreements connected with this type of problem. The first of these is that we assume that there will be stamps of each denomination in the purchase. This means that we are restricting x to be greater than 0 and y to be greater than 0.

17	In terms of the graph these are the points within the ______ quadrant.	first

Another agreement that must have been understood is that the post office stamps are sold in whole numbers; that is, x and y are restricted to whole numbers. The graph of the solution set, noting these various restrictions, are the points indicated in the graph below. The coordinates of any of these points is a possible solution. We see, for example, that Percy might buy twelve 5-cent stamps and either one or two 8-cent stamps. You might have fun in trying to answer such questions as: Which purchase leaves Percy with the least change from his dollar? Can he purchase exactly fourteen stamps? If he buys thirteen stamps, what is the least amount of change he can receive?



Consider the system $\begin{cases} 3x - 2y - 5 = 0 \\ x + 3y - 9 \leq 0 \end{cases}$.

18 This system consists of one equation and one ______. **inequality**

The truth set of the system consists of those ordered

19 pairs that satisfy the ______ and also satisfy the inequality. **equation**

Thus, the graph of the system consists of those points

20 on the line $3x$ ______ $= 0$ which lie in the region defined by $x + 3y - 9 \leq 0$. **$3x -$ [illegible] $5 = 0$**

21 Graph $\begin{cases} 3x - 2y - 5 = 0 \\ x + 3y - 9 \leq 0 \end{cases}$. **[See page 1.]**

22 Graph $\begin{cases} 4x + 2y = -1 \\ y - x \geq 4 \end{cases}$. **[See page 1.]**

23 Graph $\begin{cases} 5x + 2y + 1 > 0 \\ 3x - y - 6 = 0 \end{cases}$ **[See page 1.]**

We have dealt with systems of equations and inequalities. Each system is a compound open sentence which uses the connective "and".

Consider "$x - y - 2 > 0$ <u>or</u> $x + y - 2 > 0$".

The truth set of this sentence consists of those ordered pairs of numbers which satisfy at least one of the inequalities.

Hence, to graph "$x - y - 2 > 0$ or $x + y - 2 > 0$", we proceed in our usual way. We first graph $x - y - 2 = 0$ and shade the proper half-plane. We then graph $x + y - 2 = 0$ and again shade the proper half-plane. The graph of the compound sentences consists of all the points in either shaded region.

Graph each of the following; indicate which regions of the plane are in the truth set. Answers are on page li.

24. $x - y - 2 > 0$ or $x + y - 2 > 0$

25. $2x + y + 3 > 0$ or $3x + y + 1 < 0$

26. $2x + y + 3 \leq 0$ or $3x + y + 1 \geq 0$

Graph each of the following compound sentences. Answers are on page li.

27. $x - 3y - 6 > 0$ or $3x + y + 2 > 0$

28. $x - 3y - 6 \geq 0$ or $3x + y + 2 = 0$

29. $x - 3y - 6 < 0$ and $3x + y + 2 < 0$

30. $x - 3y - 6 > 0$ and $3x + y + 2 = 0$

The sentence $xy = 0$ does not appear to be a compound sentence.

However, our knowledge of real numbers enables us to
31 write $xy = 0$ as "$x = 0$ or ______". — $y = 0$

32 The graph of $xy = 0$ consists of ______ (how many) lines; — two

33 namely, the vertical axis, and the ______ ______. — horizontal axis

Similarly, the graph of $(2x - y + 1)(3x - 2y + 4) = 0$ consists of the two lines whose equations are
34 $2x - y + 1 = 0$ and ______. — $3x - 2y + 4 = 0$

Now consider $xy > 0$.

This is also a compound sentence, but "disguised".

For the product of two numbers to be positive, either
35 both are ______ or both are negative. — positive

Thus, $xy > 0$ is equivalent to:

36 "$x > 0$ and y _____ $y > 0$

or

37 $x < 0$ and y _____". $y < 0$

38 The graph of _____ and $y > 0$ consists of all the points in quadrant I. $x > 0$

39 The graph of $x < 0$ and _____ consists of all the
40 points in _____ III. $y < 0$ quadrant III

Hence, the graph of $xy > 0$ consists of all the points
41 in quadrants _____ and _____. I and III

Remember that quadrants do not contain the points on the axes.

Similarly,

$$(x + 2y - 4)(2x - y - 3) > 0$$

is equivalent to the compound sentence

$$\text{"}\begin{cases} x + 2y - 4 > 0 \\ 2x - y - 3 > 0 \end{cases} \quad \text{or} \quad \begin{cases} x + 2y - 4 < 0 \\ 2x - y - 3 < 0 \end{cases}\text{"}.$$

Now turn to page xlvix and examine the graph given for Item 16. The doubly shaded region is the graph of

$$\begin{cases} x + 2y - 4 > 0 \\ 2x - y - 3 > 0 \end{cases}.$$

The unshaded region is the graph of

$$\begin{cases} x + 2y - 4 < 0 \\ 2x - y - 3 < 0 \end{cases}.$$

Therefore, the graph of

$$(x + 2y - 4)(2x - y - 3) > 0$$

consists of the regions which are either doubly shaded or unshaded.

It follows that the singly shaded regions give the graph of

$$(x + 2y - 4)(2x - y - 3) < 0.$$

In practice, we proceed as in the following example.

432

To graph $(2x - y - 2)(3x + y - 3) > 0$, we first graph the two simple sentences

42 "$2x - y - 2 = 0$" and "________ $= 0$".

43 Graph these two sentences, drawing the lines with dashes.

44 The two lines divide the plane into ________ (how many) regions.

Label your graph (Item 43) as shown on page 11.

Complete the following table:

	For points in region	$2x - y - 2$ is	$3x + y - 3$ is	Product of $(2x - y - 2)(3x + y - 1)$ is
45	A	positive	positive	________
46	B	________	________	negative
47	C	________	________	positive
48	D	positive	________	________

The graph of $(2x - y - 2)(3x + y - 3) > 0$ consists

49 of regions ______ and ______.

The graph of $(2x - y - 2)(3x + y - 3) < 0$ consists

50 of regions ______.

51 Indicate by one doubly shaded and one unshaded region the graph of $(x - 3y - 6)(3x + y + 2) \leq 0$. (Begin by shading, for each factor, the region where it is positive.)

Referring to your last graph, the singly shaded

52 regions show the graph of $(x - 3y - 6)(3x + y + 2)$ ______ (<,>,=) 0.

53 Draw the graph of $(x + 2y - 6)(x + 2y + 2) > 0$ using one doubly shaded and one unshaded region. (Use suitable shadings to show first, for each factor, the region when it is positive.)

Can you find the truth set of
$(x - y - 3)(3x - 3y - 9) < 0$?

54 Before you draw a ______, think a minute! — graph

55 $3x - 3y - 9 = 3(\_\_\_\_)$. — $3(x - y - 3)$

Hence, $(x - y - 3)(3x - 3y - 9) < 0$ is equivalent to:

56 $3(\quad\quad)^2 < 0$. — $3(x - y - 3)^2 < 0$

57 The square of a real number cannot be ______. — negative

Hence, the truth set of
58 $(x - y - 3)(3x - 3y - 9) < 0$ is ______. — $\emptyset$

If you had tried to graph $(x - y - 3)(3x - 3y - 9) < 0$, you would have found that $x - y - 3 = 0$ and
59 $3x - 3y - 9 = 0$ are equations for the same ______. — line

60 This line divides the plane into ______ regions. — 2

For the points in one region, both factors of $(x - y - 3)(3x - 3y - 9)$ are positive. For the
61 other, both factors are ______. — negative

There is no point where one factor is positive and one negative.

Notice that $(x - y - 3)(3x - 3y - 9) < 0$ is equivalent to $3(x - y - 3)^2 < 0$ (See Item 56).

Since 3 is a positive number we can divide both sides by 3 and the inequality $3(x - y - 3)^2 < 0$ is in
*62 turn equivalent to ______. — $(x - y - 3)^2 < 0$

*63 Recall that by definition, $\sqrt{a^2} = \_\_\_\_$, for real numbers a. — $\sqrt{a^2} = |a|$

*64 If x, y are real numbers, $\sqrt{(x - y - 3)^2} = \_\_\_\_\_\_$, — $|x - y - 3|$
and the inequality $(x - y - 3) < 0$ leads to
*65 $|x - y - 3| < \_\_\_\_$. — $|x - y - 3| < 0$

Hence, $(x - y - 3)(3x - 3y - 9) < 0$ leads to the inequality $|x - y - 3| < 0$. The truth set of this
*66 inequality is ______. — $\emptyset$

Consider the inequality

$$|y + 3x| > 2.$$

This inequality is equivalent to

67 $(y + 3x) > 2$ $\underset{\text{(and, or)}}{\underline{\qquad\qquad}}$ $\underline{\qquad > \qquad}$.

68 Graph the sentence $|y + 3x| > 2$.

Recall that while our post office problem originally appeared to involve only the compound sentence

$$\begin{cases} x + y < 15 \\ 5x + 8y < 100 \end{cases},$$

under closer examination we also have the additional understanding that the following requirements must also be met:

$$\begin{cases} x > 0 \\ y > 0 \end{cases}, \qquad x \text{ and } y \text{ are integers.}$$

We see that our procedure may be followed even though we may have more than two clauses.

Here is a compound sentence with three clauses:

$$\begin{cases} x \geq 0 \\ y \geq 0 \\ 3x + 4y \leq 12. \end{cases}$$

69 Graph this system. (Remember the braces indicate the connective "and".

70 Graph the system

$$\begin{cases} y \geq 2 \\ 4y \leq 3x + 8 \\ 4y + 5x \leq 40. \end{cases}$$

Consider the system $\begin{cases} -4 \leq x \leq 4 \\ -3 \leq y \leq 3 \end{cases}$.

This system is equivalent to a system with four clauses, namely,

$$\begin{cases} x \geq -4 \\ \underline{\qquad\qquad} \\ \underline{\qquad\qquad} \\ y \leq 3\,. \end{cases}$$

71

72

73 Graph this system.

Compound open sentences in two variables, as we see, may involve equations, inequalities, or both. In any case, we can handle them by combining two things--our knowledge of simple open sentences in two variables and our understanding about compound sentences.

Here is an interesting, but difficult problem. It is typical of a large class of problems that arise in military planning, in manufacturing, in the transportation industry, etc.

*74 A football team finds itself on its own 40 yard line, in posession of the ball, with five minutes left in the game. The score is 3 to 0 in favor of the opposing team. The quarterback knows the team should make 3 yards on each running play, but will use 30 seconds per play. He can make 20 yards on a successful pass play, which uses 15 seconds. However, he usually completes only one pass out of three. What combination of plays will assure a victory; that is, what should be the strategy of the quarterback?

Note: Some of the assumptions we are making are simplifications of what may actually happen. For example, the assumptions that the team makes 3 yards on each running play and uses 30 seconds per play, etc., may have been estimates obtained by the past performance of the team. The combination of plays, as with other aspects of the problem, are questions in the field of probability. However, following the analysis of the solution will give us a glimpse into the character of such problems.

See the analysis and solution of this problem on page liii.

22-5. Review

The answers to the review problems are on pages liv-lv.

1. Find the truth set of "$2x - 3 = 0$" and draw its graph if it is considered as an equation in
   (a) one variable (b) two variables.

2. Find the truth set of "$|y| < 3$" and draw its graph if it is considered as a sentence in
   (a) one variable (b) two variables.

3. Given the lines with equations $3x - 5y - 4 = 0$ and $2x + 3y + 4 = 0$. Find the equations of the horizontal and vertical lines which contain the point of intersection of the two given lines.

4. Solve the following systems:

   (a) $\begin{cases} 2x = 3y + 1 \\ 4x - 3y = 11 \end{cases}$

   (b) $\begin{cases} .01x - .02y = 0 \\ x - 10y = 8 \end{cases}$

   (c) $\begin{cases} y = 2x - 4 \\ x - \frac{1}{2}y - 2 = 0 \end{cases}$

   (d) $\begin{cases} x = 9y \\ \frac{1}{3}x = 3y + 2 \end{cases}$

   (e) $\begin{cases} .5x = .2y + 4 \\ y - x = -11 \end{cases}$

   (f) $\begin{cases} \frac{1}{8}x - \frac{1}{3}y = 0 \\ \frac{1}{6}x - \frac{1}{9}y = 3 \end{cases}$

5. (a) Discuss the relationship among the coefficients of the equations of two parallel lines.

   (b) Discuss the positions of two lines if their equations are $Ax + By + C = 0$ and $Dx + Ey + F = 0$ and
   $$\frac{A}{D} = \frac{B}{E} = \frac{C}{F}.$$

   (c) Describe the conditions on the slopes of two lines which guarantee that the lines have exactly one common point.

6. Draw the graphs of the following sentences:

   (a) $y + 3x - 2 > 0$

   (b) $2x - 3y + 3 > 0$

   (c) $y + 3x - 2 > 0$, and $2x - 3y + 3 > 0$

   (d) $(y + 3x - 2)(2x - 3y + 3) < 0$

7) Find the open sentence suggested by each of the following [illegible] solve:

(a) Find two consecutive integers whose sum is [illegible].

(b) Find two integers such that their sum is [illegible] three less than the second.

(c) The sum of two numbers is 45. If the [illegible] smaller, the quotient is 4 and the remainder [illegible]. What are the numbers?

(d) Two grades of tobacco are mixed, the [illegible] and the other for \$6.00 per pound. How [illegible] must be blended to obtain 20 pounds of a mixture [illegible] per pound?

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Chapter 23

GRAPHS OF QUADRATIC POLYNOMIALS

23-1. Graphs of Equations of the Form $y = ax^2 + k$

In Chapter 17 we studied quadratic polynomials of the form $ax^2 + bx + c$, where $a \neq 0$. In this chapter, we will consider graphs of quadratic polynomials, that is, graphs of open sentences of the form

$$y = ax^2 + bx + c \ .$$

Let us begin with the polynomial x^2.

1 x^2 is a quadratic polynomial of the form $ax^2 + bx + c$, where $a =$ ____, $b =$ ____, $c =$ ____. **1, 0, 0**

2 In order to graph the quadratic polynomial x^2, we find ordered pairs of real numbers which satisfy the open sentence _____ $= x^2$. **$y = x^2$**

These ordered pairs are coordinates of points of the graph.

3 Complete the following table of ordered pairs satisfying the equation $y = x^2$. **See answer below.**

x	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3
y					0				

x	**-3**	**-2**	**-1**	$-\frac{1}{2}$	**0**	$\frac{1}{2}$	**1**	**2**	**3**
y	**9**	**4**	**1**	$\frac{1}{4}$	**0**	$\frac{1}{4}$	**1**	**4**	**9**

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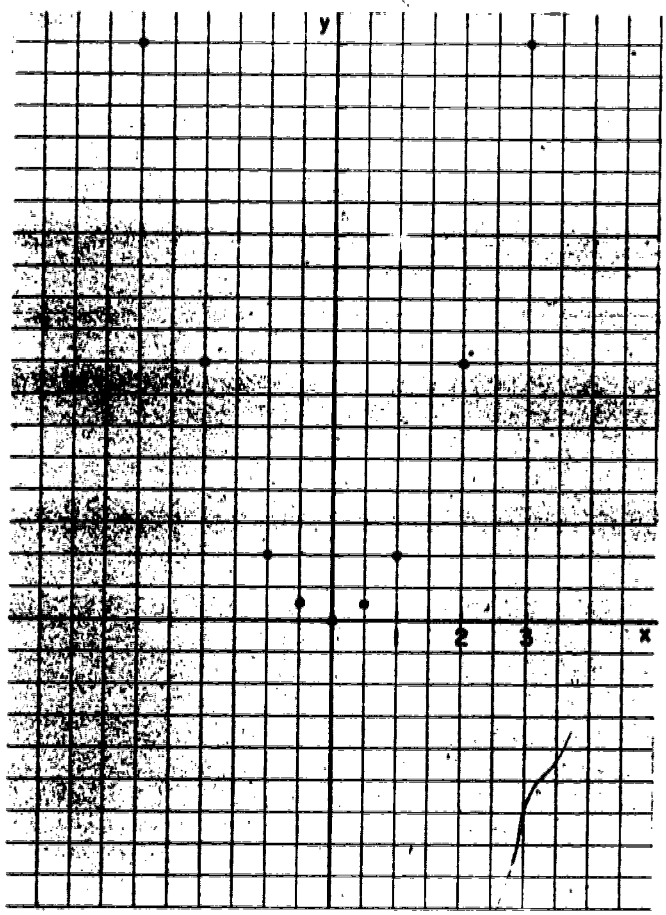
4 Locate the points corresponding to the ordered pairs indicated above with reference to a set of coordinate axes.

See answer below



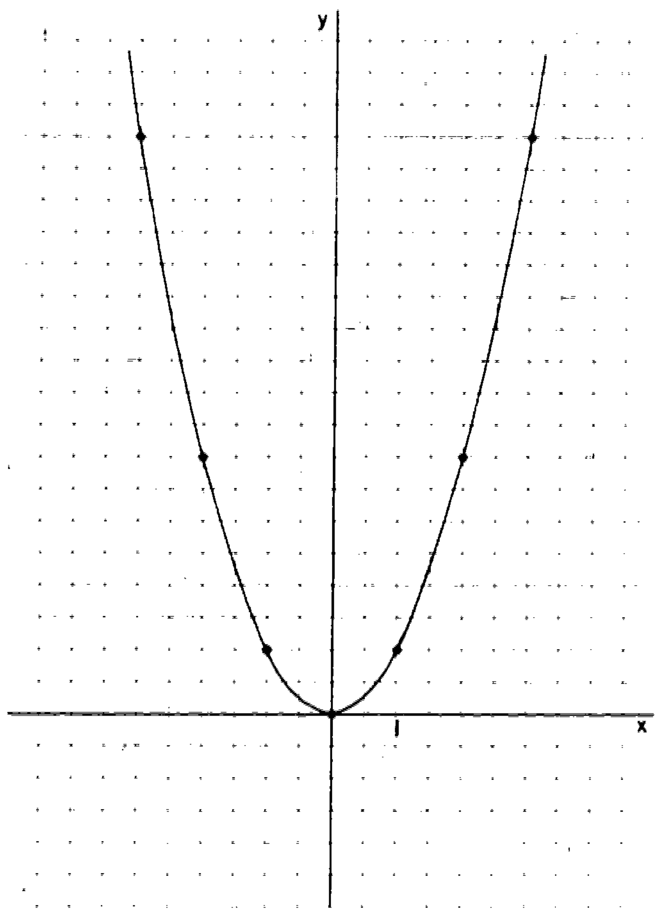
5 The arrangement of these points suggests that the graph of the equation $y = x^2$ is:

[A] A straight line.

[B] Some kind of curve different from a line.

Clearly, this is not a line. In fact, this is a curve called a "parabola". The correct choice is [B].

The graph of the equation $y = x^2$ looks like this:



The graph of $y = x^2$ passes through the point $(0, \quad)$. The value of y is always __________. (positive, negative, non-negative)	(0,0) non-negative

The graph of $y = x^2$ is called a parabola. Let us consider the graph of the equation $y = ax^2$ where $a \neq 0$.

Using the same set of coordinate axes we shall graph the equations $y = x^2$, $y = \frac{1}{2}x^2$, $y = 2x^2$. In order to do this, let us first complete the table below for $y = x^2$, $y = \frac{1}{2}x^2$, $y = 2x^2$.

See answer below.

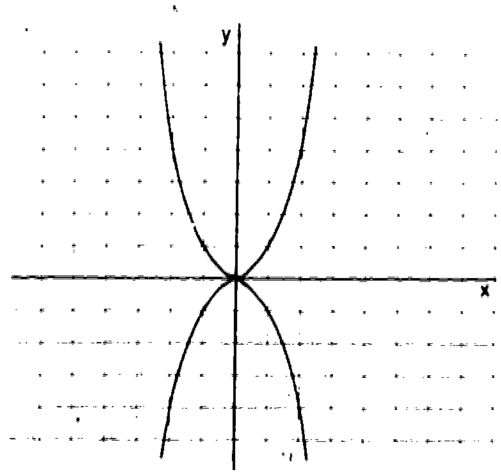
x	-4	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$		$\frac{1}{2}$	$\frac{3}{2}$	2	
x^2										
$\frac{1}{2}x^2$										
$2x^2$						0				18

x	-4	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{3}{2}$	2	3
x^2	16	4	$\frac{9}{4}$	1	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{9}{4}$	4	9
$\frac{1}{2}x^2$	8	2	$\frac{9}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	0	$\frac{1}{8}$	$\frac{9}{8}$	2	$\frac{9}{2}$
$2x^2$	32	8	$\frac{9}{2}$	2	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{9}{2}$	8	18

On the same set of coordinate axes, graph $y = x^2$, $y = \frac{1}{2}x^2$, and $y = 2x^2$. Use the table from Item 8. Compare your graphs with those on page lvi .

Consider a point on the graph of $y = x^2$. If we multiply the ordinate of this point by 2, we obtain the ordinate of a corresponding point on the graph of $y = 2x^2$. This corresponding point has the same abscissa as the point on $y = x^2$.

	Again consider a point of $y = x^2$.	
10	If we multiply the ordinate of this point by ____ we obtain the ordinate of the corresponding point on $y = \frac{1}{2}x^2$.	$\frac{1}{2}$
	Our work with the graphs of $y = x^2$, $y = 2x^2$, $y = \frac{1}{2}x^2$ suggests generalizations which apply to the graphs of equation of the form $y = ax^2$, where $a > 0$.	
11	The graph of $y = ax^2$ passes through the point $(0, \quad)$.	(0,0)
12	If $a > 0$, the ordinates of the points of the graph of $y = ax^2$ are always ________. (positive, negative, non-negative)	**non-negative**
	What is the graph of $y = -x^2$?	
13	The ordinates of the points of the graph of $y = -x^2$ are the opposites of the ______ of the corresponding points of the graph of $y = x^2$. Hence we can obtain the graph of $y = -x^2$ by <u>revolving</u> the graph of	**ordinates**
14	$y = x^2$ one half revolution about the ________. (y-axis, x-axis)	**x-axis**
15	Below are graphs of $y =$ ______ and $y =$ ______.	$y=x^2$ **and** $y=-x^2$



16 The graphs of $y = x^2$ and $y = -x^2$ both contain the point $(\_\_,\_\_)$. — **(0,0)**

17 The ordinates of the points on the graph of $y = x^2$ are always non-negative and the ordinates of the points of the graph of $y = -x^2$ are always ______. — **non-positive**

18 In the same manner, we could find the graph of $y = -3x^2$ by revolving the graph of $y =$ _____ one-half revolution about the x-axis. — $y = 3x^2$

19 The graph of $y = \frac{1}{3}x^2$ may be obtained by revolving the graph of $y =$ _____ one-half revolution about the x-axis. — $y = -\frac{1}{3}x^2$

20 Graph $y = 3x^2$, $y = -3x^2$, $y = \frac{1}{3}x^2$ and $y = -\frac{1}{3}x^2$ on the same coordinate axes.

Turn to page lvii to check your graphs.

21 How can we obtain the graph of $y = -ax^2$ from the graph of $y = ax^2$, where a is any non-zero real number?

P. By revolving the graph of $y = ax^2$ one-half revolution about the x-axis.

Q. By taking the opposite of the ordinate of each point on the graph of $y = ax^2$ to get the ordinate of the point with the same abscissa on the graph of $y = -ax^2$.

[A] only P [B] only Q [C] P or Q

[C] is correct. If we take the opposite of the ordinate of a point of the graph of $y = ax^2$ we get the ordinate of the corresponding point of the graph of $y = -ax^2$. This is the same as revolving the graph of $y = ax^2$ one-half revolution about the x-axis.

The three curves in Item 20 are all parabolas. Parabolas occur in many applications of mathematics. Some telescopes have parabolic lenses. A bullet fired from a gun travels in a path which is approximately parabolic. Since parabolas have many interesting properties and useful applications, they are worth studying carefully.

Let us look once more at the graph of $y = x^2$.

	Among the points on the graph of $y = x^2$ are:	
22	$(\underline{\quad}, 1)$ and $(-1, 1)$	$(1, 1)$
23	$(2, 4)$ and $(\underline{\quad}, 4)$	$(-2, 4)$
24	$(3, 9)$ and $(\underline{\quad}, 9)$	$(-3, 9)$
25	$(\frac{1}{2}, \frac{1}{4})$ and $(-\frac{1}{2}, \underline{\quad})$	$(-\frac{1}{2}, \frac{1}{4})$
26	In fact, if (p, q) is on the graph of $y = x^2$, then $(-p, \underline{\quad})$ is also a point on the graph of $y = x^2$.	[illegible]
27	This is clear from the equation $y = x^2$. For if $q = p^2$ is a true sentence, then $q = (-p)^2$ is also a ______ sentence.	true
	The points (p, q) and $(-p, q)$ are the same distance from the horizontal axis and on the same side of it.	
28	The points $(\underline{\quad}, q)$ and $(-p, q)$ are also the same distance from the vertical axis, but the two points	(p, q)
29	lie on __________ (opposite, the same) side(s) of this axis.	opposite
30	If the number plane were folded precisely along the line $x = \underline{\quad}$, the parts of the graph of $y = x^2$ on either side of the line $x = 0$ would coincide.	$x = 0$
31	For example, the points $(\frac{5}{2}, \frac{25}{4})$ and $(\underline{\quad}, \underline{\quad})$ would be together after the number plane was folded.	$(-\frac{5}{2}, \frac{25}{4})$
	We observe that the graph is symmetric about the line $x = 0$.	
32	The line $x = 0$ is also called the ___-axis.	y-axis
33	We say that the graph of $y = x^2$ is ______ about the y-axis.	symmetric

The line (in this case the y-axis) about which a parabola is symmetric is called the <u>axis</u> of the parabola. The point where the parabola intersects its axis is called the <u>vertex</u> of the parabola.

For the parabola whose equation is $y = x^2$, the axis	
34 is the line $x =$ ___. The vertex has coordinates	$x = 0$
35 _____.	(0, 0)

Refer, if necessary, to the graphs you have drawn in this section to complete the following items.

The graphs of $y = x^2$, $y = \frac{1}{2}x^2$, $y = 3x^2$, $y = -3x^2$	
36 all pass through the point (__ , __).	(0, 0)
The axis of each of these parabolas is the line	
37 ___ = ___, and the vertex of each is	$x = 0$
38 (__ , __).	(0, 0)
Indeed, for every non-zero real number a, the graph	
39 of $y = ax^2$ is a ______ whose axis is the y-axis	**parabola**
40 and whose vertex is (__ , __).	(0, 0)
41 Moreover, if $a > 0$ the vertex is the ________ (highest, lowest) point on the curve. In this case we sometimes say the curve opens upward.	**lowest**
42 If $a < 0$, the vertex is the ________ (highest, lowest) point point on the curve. We might say that in this case the curve opens ________.	**highest**

Now that we know about the graphs of quadratic polynomials which can be expressed in the form ax^2, let us investigate the graphs of polynomials of the form $ax^2 + k$. As usual, we will start with a particular example.

How does the graph of $y = x^2 + 3$ compare with the graph of $y = x^2$?

Complete the following table.

43

x	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3
x^2									
$x^2 + 3$									

See answer below.

x	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3
x^2	9	4	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	4	9
$x^2 + 3$	12	7	4	$\frac{13}{4}$	3	$\frac{13}{4}$	4	7	12

44 Graph $y = x^2 + 3$ and $y = x^2$ using the same coordinate axes. Turn to page lvii to check your graphs.

Use the graphs in Item 44 to complete the following items.

45 The graph of $y = x^2 + 3$ is symmetric about the ______-axis. The y-axis is the axis of the parabola. **y-axis**

46 The graph of $y = x^2 + 3$ intersects the y-axis at the point (___ , ___). The point $(0,3)$ is the **(0,3)**

47 ______ of the parabola. **vertex**

48 The point $(0,3)$ is the ________ (highest, lowest) point on the graph of $y = x^2 + 3$. **lowest**

The graph of $y = x^2 + 3$ may be obtained from the graph of $y = x^2$ by moving the graph of $y = x^2$

49 ________ (upward, downward) 3 units. **upward**

Items 45 to 49 described graphs you had drawn. When you drew these graphs you located a few points and drew a smooth curve through them. You assumed that all of the points corresponding to ordered pairs satisfying $y = x^2$ lie on the curve you drew for $y = x^2$ (or on the continuation of it beyond your drawing.) You also assumed that every point on this curve has coordinates which satisfy the equation. We have made similar assumptions about the other graphs we have drawn.

A complete justification of these assumptions is beyond the scope of this course. However, we <u>can</u> show that certain properties of the curves drawn for $y = x^2$ and $y = x^2 + 3$ follow from algebraic properties of the equations. The relation between geometric properties of curves and algebraic properties of equations makes algebra useful for studying curves such as parabolas.

We can prove that the lowest point on the graph of $y = x^2 + 3$ is the point with ordinate 3; that is,

50 the point (,).

All we need to remember is that if x is any real number then x^2 is non-negative.

From this it follows that if x is any real number

51 $x^2 + 3$ is at least ____.

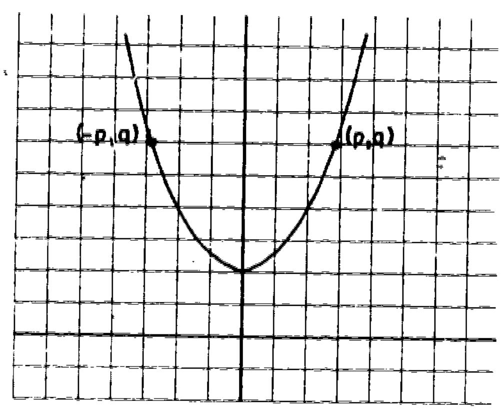
All the points on the graph of $y = x^2 + 3$ except

52 (,) have ordinates greater than 3.

3

(0,3)

We can also show easily that the graph of $y = x^2 + 3$ is symmetric about the y-axis. Symmetry around the y-axis means: For each point on one side of the y-axis, there is a point on the other side, with the same ordinate, such that the two points are the same distance from the y-axis.



Thus in proving that the graph of $y = x^2 + 3$ is symmetric about the y-axis, we need only show that--given any point on the curve--the point with the same ordinate and opposite abscissa also lies on the curve.

We suppose then, that (p,q) is a point on the graph

53 of $y = x^2 + 3$. We must show that (, q) is also a point of the graph.

If a point is on the graph of $y = x^2 + 3$, then its coordinates satisfy the equation.

Hence, if (p,q) is on the graph of $y = x^2 + 3$, we see that p and q are real numbers such that

54 q = ______.

(-p,q)

$q = p^2 + 3$

55	In order to show that $(-p,q)$ lies on the graph, we must show that ______ $= (-p)^2 + 3$ is true. We can do this by using Item 54 and the fact that $(-p)^2 = p^2$ for all real values of p.	$q = (-p)^2 + 3$.

Finally, let us interpret in terms of coordinates the idea of "moving the graph of $y = x^2$ upward 3 units".

56	For any value of x, the value of $x^2 + 3$ is 3 ______ (greater, less) than the value of x^2.	greater
57	Thus for points on the graphs of $y = x^2$ and $y = x^2 + 3$ whose abscissas are equal, the ordinate of the point on the graph of $y = x^2 + 3$ is ______ greater than the ordinate of the corresponding point on the graph of $y = x^2$.	3
58	That is, if (p,q) is on the graph of $y = x^2$, then $(p, \_\_\_)$ is on the graph of $x^2 + 3$.	$(p, q + 3)$
59	This means that in drawing the graph of $x^2 + 3$ we can think of moving the graph of x^2 upward ____ units.	3
60	For example, the point $(0,3)$ on the graph of $y = x^2 + 3$ is 3 units above its corresponding point $(\_\_ , \_\_)$ on the graph of $y = x^2$. The shapes of the two graphs are alike.	$(0,0)$

Let us look at another example.

61	Graph $y = x^2$ and $y = x^2 - 3$ on the same set of coordinate axes. Turn to page lviii to check your graphs.	
62	The vertex of the graph of $y = x^2 - 3$ is ______.	$(0,-3)$
63	The curve opens ______ (upward, downward).	upward
64	The graph of $y = x^2 - 3$ may be obtained by moving the graph of $y = x^2$ ______ (upward, downward) 3 units.	downward

65	The graph of $y = x^2 - 3$ intersects the x-axis at two points, _______ and _______. Notice that in the last item we found the ordinates of the points by finding the solutions of an equation in one variable, $x^2 - 3 = 0$.	$(0,\sqrt{3})$ [illegible]

Graph the following equations. Compare your graphs with those on page lviii.

66 $y = x^2 - 2$

67 $y = \frac{1}{2}x^2$

Do these on the same set of coordinate axes.

68 $y = 3x^2$

69 $y = 3x^2 + 2$

70 $y = 3x^2 - 2$

Do these on the same set of coordinate axes.

71 $y = -x^2$

72 $y = -x^2 + 3$

73 $y = -x^2 - 2$

Do these on the same set of coordinate axes.

Look back at your graphs in Items 66 to 73. In each case we have an equation of the form $y = ax^2 + k$, where $a \neq 0$.

	Our examples illustrate the fact that:	
74	The vertex of the parabola $y = ax^2 + k$ is the point $(\_\_,\_\_)$.	$(0,k)$
75	The parabola $y = ax^2 + k$ opens upward if $a > 0$. If $a < 0$, then the parabola opens _______.	downward
76	The equation $y = -x^2 + 3$ is of the form $y = ax^2 + k$, with $a =$ ___, $k =$ ___.	[illegible]
77	The parabola $y = -x^2 + 3$ has vertex $(\_\_,\_\_)$.	[illegible]
78	It opens _______.	[illegible]
79	The vertex of the parabola $y = -x^2 - 2$ is $(\_\_,\_\_)$.	$(0,-2)$

23-2. Graphs of Equations of the Form $y = a(x - h)^2 + k$

Now that we know something about the graphs of quadratic polynomials which can be expressed in the form $ax^2 + k$, let us proceed to other types of quadratic polynomials. We wish to build on what we have already discovered. It is reasonable, therefore, to treat a polynomial which has some similarity to one with which we are now familiar.

Let us compare the graph of $y = (x - 3)^2$ with that of $y = x^2$. In order to do this, let us first complete the table of values below.

1

x	-2	-1	0	1	2	3	4	5	6
x^2	4								
$(x-3)^2$	25								

See answer below.

x	-2	-1	0	1	2	3	4	5	6
x^2	4	1	0	1	4	9	16	25	36
$(x-3)^2$	25	16	9	4	1	0	1	4	9

2 On the same set of coordinate axes, graph $y = x^2$ and $y = (x - 3)^2$. Compare your graphs with those on page 1x .

If you compare the second and third rows in your table of values (Item 1), you will notice that the third row and the second both exhibit the sequence of

3 numbers 4, 1, 0, 1, ___, ___. — 4, 9

4 However, this sequence appears ______ spaces later in the third row than in the second. — three

Now examine the graphs which you have just drawn. The two graphs seem to have the same shape. One appears to be obtained from the other by a shift sideways. Let us make these ideas more precise.

It appears that to every point on the graph of $y = x^2$ there corresponds a point three units to the right on the graph of $y = (x - 3)^2$. These two points have the same ordinate.

5. For example, $(2,4)$ is on the graph of $y = x^2$ and $(2 + \_\_, 4)$ is on the graph of $y = (x - 3)^2$. $(2 + 3, 4)$

6. If (p,q) is any point, the point which is three units to the right of it and has the same ordinate is $(\_\_, \_\_)$. $(p + 3, q)$

It appears from the graphs of $y = x^2$ and $y = (x - 3)^2$ if (p,q) is any point on the graph of $y = x^2$, then

7. $(p + 3, q)$ is a point on the graph of $y = \_\_\_\_$. $y = (x - 3)^2$

It is easy to prove that this is indeed the case.

Suppose (p,q) is on the graph of $y = x^2$. Then the coordinates satisfy the equation $y = x^2$.

8. That is, $q = \_\_\_\_$ is a true sentence. $q = p^2$

In order to show that $(p + 3, q)$ is on the graph

9. of $y = (x - 3)^2$, we must show that $\_\_ = ((p+3) - 3)^2$ is also a true sentence. $q = ((p+3)-3)^2$

We can do this easily, using Item 8 and the fact that

10. if p is any real number $(p + 3) - 3 = \_\_$. p

Let us apply our findings to the graphs of these two quadratic polynomials. A point in the number plane belongs to the graph of an open sentence in two variables if and only if its coordinates are an ordered pair belonging to the truth set of the open sentence.

We have shown that if (p,q) is a point on the parabola $y = x^2$ then $(p + 3, q)$ is a point on the parabola $y = (x - 3)^2$. We can think of the point $(p + 3, q)$ as being obtained by moving the point (p,q) to the right 3 units. Thus the graphs of $y = (x - 3)^2$ and $y = x^2$ have the same shape.

11. The vertex of the parabola $y = x^2$ is $(\_\_\_)$. $(0,0)$

12. The vertex of the parabola $y = (x - 3)^2$ is $(\_\_, \_\_)$. $(3,0)$

Note that the points of the graph of $y = (x - 3)^2$ have non-negative ordinates, since $(x - 3)^2$ is

13. non-____ for all real values of x. non-negative

Using reasoning similar to that used above we could show:	
To obtain the graph of $y = 2(x - 5)^2$, we could move the graph of $y = 2x^2$ ______ units to the ______ (left, right).	5 right
Draw the graph of $y = 2x^2$ and the graph of $y = 2(x - 5)^2$, using the same pair of coordinate axes. Compare your graphs with those on page 1x .	
To obtain the graph of $y = -2(x - 3)^2$, we could move the graph of $y = -2x^2$ to the ______ 3 units.	right
The vertex of the parabola $y = -2(x - 3)^2$ is ______.	(3,0)
The vertex is the ______ (highest, lowest) point on the graph of $y = -2(x - 3)^2$.	highest
Since $-2(x - 3)^2$ is negative for all real values of x except ______.	3

On graph paper draw the graph of $y = (x + 2)^2$. Then complete the following.

The graph of $y = (x + 2)^2$ is a parabola which opens ______ (upward, downward).	upward
The vertex, (______), is the lowest point.	(-2,0)
We obtain the graph of $y = (x + 2)^2$ if we move the graph of $y = x^2$ to the ______ 2 units.	left
The axis of the parabola $y = (x + 2)^2$ is the line ______.	x = -2

As a generalization of these results we may state:

25 The graph of the equation $y = a(x - h)^2$ is a ______ — parabola
26 whose vertex is the point (,). — (h,0)

The graph of $y = a(x - h)^2$ may be obtained by
27 moving the graph of ______ horizontally a — $y = ax^2$
28 distance of | | units. — $|h|$

To illustrate these statements, let us consider $y = 2(x + 2)^2$.

29 Since $x + 2 = x - (\ \ )$, we see that, in — -2
30 $y = 2(x + 2)^2$, we have $a = 2$ and $h =$ ____. — $h = -2$

The graph of $y = 2(x + 2)^2$ is a parabola with its
31 vertex at (,). — (-2,0)

32 Its axis is $x =$ ____. — $x = -2$

33 We move the graph of $y = 2x^2$ to the ______ 2 — left
units to obtain the graph of $y = 2(x + 2)^2$.

We have seen how the graphs of $ax^2 + k$ and $a(x - h)^2$ are related to that of ax^2. Now we are ready to consider the graph of $a(x - h)^2 + k$. Try to decide how the graph of $a(x - h)^2 + k$ is related to that of ax^2. Then compare your conclusions with those in the following items.

Consider the graph of $y = \frac{1}{2}(x - 3)^2 + 2$.

34 Our experience suggests that this graph is a ______. — parabola

Moreover, we would expect that we obtain the parabola
35 by moving the graph of $y =$ ___ x^2 to a new — $y = \frac{1}{2}x^2$
position.

The graph of $y = \frac{1}{2}x^2$ is a parabola which opens
36 ______. — upward
(upward, downward)

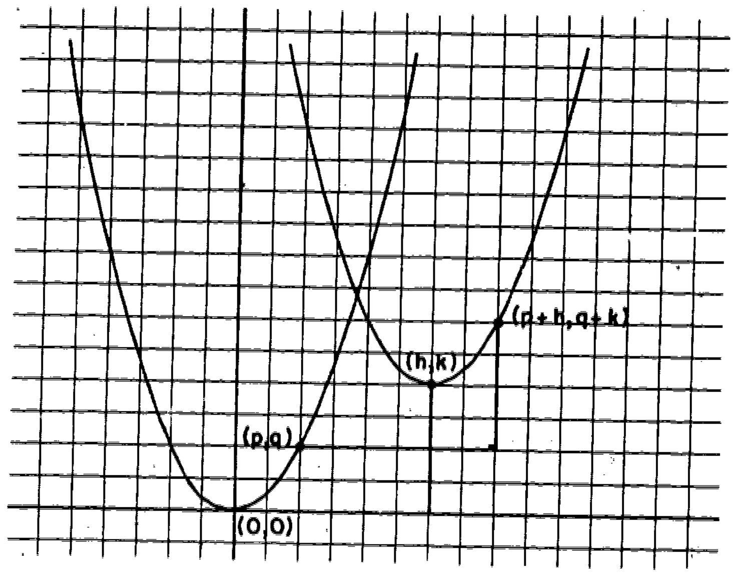
37 The vertex of the parabola, $y = \frac{1}{2}x^2$ is (,), — (0,0)
which is the lowest point on the curve.

On the graph of $y = \frac{1}{2}(x - 3)^2 + 2$, the ordinate of
38 every point is at least ____. — 2

Item	Frame	Answer
39	If x is 3, $(x - 3)^2$ is ____.	0
40	Hence, the lowest point on the curve is the point with ordinate 2; that is, the point (,).	(3,2)
41	This point is the ______ of the parabola $y = \frac{1}{2}(x - 3)^2 + 2$.	vertex
42	The parabola $y = \frac{1}{2}(x - 3)^2 + 2$ is obtained by moving the parabola $y = \frac{1}{2}x^2$ to the right ____ units	3
43	and $\underset{\text{(up, down)}}{\underline{\hspace{4em}}}$ 2 units.	up

The conclusions in Items 34 to 43 are suggested by our experience with several examples. You will probably find it convenient to use this reasoning in drawing graphs. If you wish to see a more precise statement of the reasoning we have used complete Items *44 to *48. If not go to Item 49.

Item	Frame	Answer
*	Consider the sentences $y = ax^2$ and $y = a(x - h)^2 + k$.	
*44	Let (p,q) be coordinates of a point on the graph of $y = ax^2$. Then $q =$ ____ is true.	$q = ap^2$
	Then we can show that point $(p + h, q + k)$ lies on the graph of $y = a(x - h)^2 + k$.	
*45	We need only note that $$q + \underline{\hspace{3em}} = a((p + h) - h)^2 + k$$ is true.	$q + k$ (See Item *44.)
*46	To every point (p,q) on $y = ax^2$ there corresponds the point $(p + h, \underline{\hspace{2em}})$ on $y = a(x - h)^2 + k$.	$(p + h, q + k)$
*47	In particular, we see that the vertex, (,) of the parabola $y = ax^2$ corresponds in this way to the	(0,0)
*48	vertex, (,), of the parabola $y = a(x - h)^2 + k$. The graphs below should help you understand the situation.	(h,k)



Suppose we wish to draw the graph of $y = -(x - 3)^2 - 4$. We might begin by thinking about a simpler graph,
49 that of $y = -x^2$. This graph, we know, opens _______.
Now returning to $y = -(x - 3)^2 - 4$, we note that all
50 the points on this curve except $(\ \ ,-4)$ have ordinates less than -4.
51 Draw the graphs of $y = -x^2$ and $y = -(x - 3)^2 - 4$ on the same axes. Compare your graph with the one on page lxi .

We have now discussed thoroughly the graphs of quadratic polynomials of the form $a(x - h)^2 + k$. Every such polynomial has a graph which is a parabola and is simply related to the graph of the polynomial ax^2. Given a polynomial $a(x - h)^2 + k$, you should be able to draw its graph with a minimal amount of work. You should be able to "read off" the coordinates of the vertex of the parabola which $y = a(x - h)^2 + k$ represents. In drawing graphs you should remember that every point on the graph has coordinates which satisfy the equation. If you are uncertain, it is wise to make a brief table of values.

Answers to the following problems are on pages lxi - lxiii.

52. Which of the following have graphs which can be obtained by moving the graph of "$y = 2(x + 3)^2 - 6$" to another position?

(a) $y = 2x^2$ (c) $y = 2(x + 3)^2 + 6$ (e) $y = 2(x - 10)^2$

(b) $y = 3(x + 3)^2 - 6$ (d) $y = 2(x - 12)^2 + 157$ (f) $y = (x + 3)^2 - 6$

53. Which of the numbers a, h, k in the polynomial $a(x - h)^2 + k$ determines the "shape" of the graph of the polynomial? That is, which number determines how rapidly it spreads out?

54. Describe how the graph of $y = x^2 - 2$ and the graph of $y = x^2 + 2$ can be obtained from the graph of $y = x^2$. Draw all three graphs with reference to the same set of axes.

55. How can the graph of $y = 2(x - 2)^2 + 3$ be obtained from the graph of $y = 2(x - 2)^2$? Draw both graphs with reference to the same set of axes.

56. How is the graph of $y = -2(x + \frac{1}{2})^2 + 3$ obtained from the graph of $y = -2x^2$? Draw both graphs with reference to the same set of axes.

57. Without drawing the graphs, describe the graph of each of the following by telling how it can be obtained from the graph of some polynomial of the form ax^2.

(a) $y = 3(x - 7)^2 - \frac{1}{2}$
(b) $y = 3(x - \frac{1}{2})^2 + 7$
(c) $y = 2x^2 + \frac{5}{2}$
(d) $y = 2(x + \frac{5}{2})^2$
(e) $y = -(x + 3)^2 - 4$
(f) $y = x^2 + 14$
(g) $y = 5x^2 + 9$
(h) $y = 5(x - 2)^2 + \frac{7}{2}$
(i) $y = -8(x - 8)^2 - 8$
*(j) $y = 4(3 - x)^2 - 6$

58. Draw the graph of x^2 for x such that $-2 \leq x \leq 2$. Then draw the graph of $5x^2$ on the same set of axes.

59. Give the coordinates of the vertex and the equation of the axis of each of the following parabolas.

(a) $y = x^2$
(b) $y = 5x^2$
(c) $y = -5x^2$
(d) $y = 5(x - 2)^2$
(e) $y = 5(x - 2)^2 + 3$
(f) $y = 5(x - 2)^2 - 3$
(g) $y = 5(x + 2)^2$
(h) $y = 5(x + 2)^2 + \frac{1}{2}$
(i) $y = 5(x + 2)^2 - \frac{1}{2}$

23-3. Quadratic Polynomials of the Form $y = ax^2 + bx + c$

We have seen that the graph of every polynomial of the form $a(x - h)^2 + k$ is a parabola.

	Now let us consider the polynomial $x^2 - 2x - 3$.	
1	This ______ (is, is not) a polynomial of the form $a(x - h)^2 + k$.	is not
	However, this fact need not alarm us.	
2	$x^2 - 2x - 3 = \underline{(x^2 - 2x + \quad) - \quad - 3}$	$(x^2 - 2x + 1) - 1 - 3$
3	$= \underline{(\quad)^2 - 4}$	$(x - 1)^2 - 4$
4	In Items 2 and 3 we have used the method of completing the ______ to rewrite the polynomial $x^2 - 2x - 3$ as $(x - 1)^2 - 4$.	completing the square
5	We call $(x - 1)^2 - 4$ a quadratic polynomial in ______ form.	standard form

If we wish to draw the graph of $y = x^2 - 2x - 3$, we can begin by writing the equivalent equation $y = (x - 1)^2 - 4$. This equation is of the form $y = (x - h)^2 + k$. Since an equation in this form is so convienient for studying the parabola, we call it the standard form of the equation of the parabola.

In Chapter 17 we saw that every quadratic polynomial -- that is, every polynomial of form $ax^2 + bx + c$, where $a \neq 0$, can be written in standard form.

	Since every quadratic polynomial can be written in standard form, every equation of the form $y = ax^2 + bx + c$, where $a \neq 0$, can be written in the form $y = a(x - h)^2 + k$.	
6	We conclude: the graph of every equation of the form $y = ax^2 + bx + c$, where $a \neq 0$, is a ______.	parabola

453

In order to graph $y = x^2 + 2x + 3$, we can first
7 write the equivalent equation $y = \underline{(\quad\quad)^2 + 2}$. $(x + 1)^2 + 2$

The vertex of the parabola $y = (x + 1)^2 + 2$
8 is $\underline{(\quad , \quad)}$. $(-1,2)$

The equation in standard form of the parabola
9 $y = x^2 - 10x + 3$ is $\underline{y = (\quad\quad)^2\quad\quad}$. $y = (x - 5)^2 - 22$

If you had difficulty with these items you should review Section 16-5 and Section 17-3, where completing the square is discussed in detail.

Write the equation of each parabola in standard form. If you have difficulty, refer to the items noted in Chapter 17, Section 3.

10 $y = 2x^2 + 8x + 3$ $\quad \underline{y = \quad\quad\quad\quad}$ $y = 2(x+2)^2 - 5$
(Items 55 to 59, Section 17-3.)

11 $y = -x^2 - 2x + 3$ $\quad \underline{y = \quad\quad\quad\quad}$ $y = -(x+1)^2 + 4$
(Items 71 to 73, Section 17-3.)

12 $y = 3x^2 - 4x + 5$ $\quad \underline{y = \quad\quad\quad\quad}$ $y = 3(x - \frac{2}{3})^2 + \frac{11}{3}$
(Items 75 to 76, Section 17-3.)

We have already solved quadratic equations in one variable. These equations have the form $ax^2 + bx + c = 0$.

In order to solve $x^2 + 8x - 2 = 0$, we might write the following chain of equivalent equations:

13 $x^2 + 8x \underline{+ \quad\quad - \quad\quad} - 2 = 0$ $\quad + 16 - 16$

14 $(x + 4)^2 - \underline{\quad\quad} = 0$ $\quad 18$

15 $(x + 4 + \sqrt{18})\underline{(\quad\quad\quad)} = 0$ $\quad (x + 4 - \sqrt{18})$

16 $x = -4 - \sqrt{18}$ or $\underline{x = \quad\quad\quad}$ $\quad x = -4 + \sqrt{18}$

17 The truth set of the equation is $\underline{\{\quad\quad\quad\quad\}}$. $\quad \{-4-\sqrt{18}, -4+\sqrt{18}\}$

Let us consider now the graph of the equation in two variables

$$y = x^2 + 8x - 2.$$

In order to graph $y = x^2 + 8x - 2$, we can first write the equation of this parabola in standard form.

Thus we write: $y = (x + 4)^2 - 18$

(Compare with Item 14 above.)

18 The parabola $y = (x + 4)^2 - 18$ has vertex (,) (-4,-18)

19 and opens ______ (upward,downward). upward

Notice that the vertex is <u>below</u> the x-axis, and that the parabola opens upward.

We would expect, therefore, that the parabola inter-

20 sects the x-axis in ______ (how many) points. (Draw a sketch if you aren't sure.) two

21 For every point on the x-axis, the ______ (abscissa, ordinate) ordinate

is 0. Consequently, the points where the parabola crosses the x-axis are the points on the parabola with ordinate 0.

If the ordinate of a point on the parabola $y = (x + 4)^2 - 18$ has ordinate (y value) 0, then its abscissa (x value) is an element of the truth

22 set of $(x + 4)^2 - 18 =$ ____. 0

This is the quadratic equation we solved above. Its

23 truth set is ______. (See Item 17.) $\{-4-\sqrt{18}, -4+\sqrt{18}\}$

Hence the graph of $y = (x + 4)^2 - 18$ intersects the

24 x-axis in the points ( ), ( ). $(-4-\sqrt{18}, 0)$; $(-4+\sqrt{18}, 0)$

We may summarize Items 18 to 24 as follows:

$y = x^2 + 8x - 2$ is an equation in two variables,

25 ____ and ____. x, y

26 Its graph is a ____. parabola

4_0

27 This parabola intersects the x-axis in ______ (how many) — two

28 points. The ordinate of each of these points is ____. — 0

The abscissas of these points are found by solving

29 the ______ equation — quadratic

30 $x^2 + 8x$ ______________, — $x^2 + 8x - 2 = 0$

regarding it as an equation in the single variable x.

31 The truth set of this equation has ________ (how many) elements. — two

---

Consider the graph of $y = x^2 - 4x + 4$. Draw the graph on scratch paper. Then complete the following items.

When we write this equation of a parabola in standard

32 form we have ______. — $y = (x - 2)^2$

33 The vertex of the parabola is (__ , __). The — (2,0)

34 vertex is ______________ (on, above, below) the x-axis. — on

Notice that the quadratic equation in one variable

35 $x^2 - 4x + 4 = 0$ has truth set { ____ }. — {2}

Draw the graph of $y = -x^2 + 6x - 10$ on scratch paper. Then complete the following items.

36 The standard form of this equation is ______. — $y = -(x-3)^2 - 1$

37 This vertex of this parabola is (__ , __), and — (3,-1)

38 the parabola opens ______. — downward

39 The curve ______________ (does, does not) cross the x-axis. — does not

Hence the truth set of the quadratic equation in one

40 variable $-x^2 + 6x - 10 = 0$ is ______. — ϕ

$-x^2 + 6x - 10 = -(x - 3)^2 - 1$.

$-(x - 3)^2 - 1$ is negative for all real values of x. This is consistent with our conclusion, in Item 40, that $-x^2 + 6x - 10 = 0$ has the truth set ϕ.

*	Consider the equation $y = ax^2 + bx + c$. We will write an equivalent equation which is in standard form. $y = a(x^2 + \frac{b}{a}x) + c$		
*41	$y = a\left(x^2 + \frac{b}{a}x + (\quad)^2\right) - a(\frac{b}{2a})^2 + c$ [Note that adding $(\frac{b}{2a})^2$ inside the parentheses requires us to subtract $a(\frac{b}{2a})^2$.]		$(\frac{b}{2a})^2$
*42	Hence we have: $y = a(x + \quad)^2 - \frac{b^2 - 4ac}{4a}$		$a(x + \frac{b}{2a})^2$
*43	The graph of this equation is a parabola with vertex at $(\quad,\quad)$.		$(-\frac{b}{2a}, -\frac{b^2-4ac}{4a})$

* Items *41 to *43 should remind you of the derivation of the quadratic formula in Section 17-4. This derivation is summarized below. If you have trouble following it, refer to Section 17-4.

* We found (Item 43) that the equation $y = ax^2 + bx + c$ is equivalent to

$$y = a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a} .$$

In order to find the points where the curve crosses the x-axis, we must solve the quadratic equation in one variable.

$$a(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a} = 0 .$$

An equivalent equation is

$$(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a^2} = 0 .$$

We note: $4a^2 > 0$ for all real values of a different from 0. If $b^2 - 4ac > 0$, then the left side is the difference of two squares. Hence an equivalent open sentence is:

$$(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a})(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}) = 0$$

Thus if $b^2 - 4ac > 0$, then the truth set of this equation, and hence of the equivalent equation $ax^2 + bx + c = 0$, is

$$\{\frac{-b - \sqrt{b^2 - 4ac}}{2a}, \frac{-b + \sqrt{b^2 - 4ac}}{2a}\} .$$

If $b^2 - 4ac = 0$, then the truth set is $\{-\frac{b}{2a}\}$.

If $b^2 - 4ac < 0$, then the truth set is $\emptyset$.

Solve, using the quadratic formula:

*44	$x^2 - 10x + 21 = 0$	______	$\{7,3\}$
*45	Hint: $a =$ ___, $b =$ ___, $c =$ ___.		1, -10, 21
*46	$x^2 = 2x + 1$	______	$\{1+\sqrt{2}, 1-\sqrt{2}\}$
	Hint: first write an equivalent equation in the form $ax^2 + bx + c = 0$.		
*47	$x + 6x^2 = 1$	______	$\{\frac{1}{3}, -\frac{1}{2}\}$
*48	$x^2 + 1 = 4x$	______	$\{2+\sqrt{3}, 2-\sqrt{3}\}$

* Consider the graph of $y = ax^2 + bx + c$.

We have seen (Item *43) that its vertex is $\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$.

If $b^2 - 4ac < 0$ and $a < 0$ then the vertex is

*49 ______ (above, below) the x-axis. Also the curve opens — below

*50 ______ (upward, downward). — downward $[a < 0]$

*51 In this case the parabola ______ (does, does not) cross the x-axis. — does not

You can verify for the other possibilities, in a similar way, that if $b^2 - 4ac < 0$ the parabola does not cross the x-axis; that if $b^2 - 4ac > 0$ the parabola crosses the x-axis in two points; and that if $b^2 - 4ac = 0$ the vertex of the parabola is on the x-axis.

23-4. Summary and Review

We have learned that the graph of every equation of the form $y = a(x - h)^2 + k$ is a parabola.

The vertex of the parabola is the point (h,k).

The axis is the line $x = h$.

The parabola opens upward if $a > 0$ and downward if $a < 0$.

Every polynomial of form $ax^2 + bx + c$ can be expressed as a polynomial in standard form $a(x - h)^2 + k$. It follows that every equation of the form $y = ax^2 + bx + c$ can be written in the form $y = a(x - h)^2 + k$, which is called the standard form of the equation of the parabola.

Twelve open sentences are listed below. Each of the twelve has a graph which may be labeled either "parabola," "line," "pair of lines," or "point."

Identify each graph with the one label that is correct.

1	$y = x$	______	line
2	$y = x^2$	______	parabola
3	$y^2 = x^2$	______	pair of lines
4	$y^2 = x^2$ is a pair of lines because $y^2 = x^2$ is equivalent to $(y + x)(\_\_\_\_\_) = 0$.		$y - x$
5	$y = 0 \cdot x^2$	______	line
6	$x = 4$	______	line
7	$x^2 - y = 4$	______	parabola
8	$x - y = 4$	______	line
9	$y = (x - 3)^2$	______	parabola
10	$0 = (x - 3)^2$	______	line
11	"$x - 3 = y$ and $2x - y = 3$"	______	point
12	"$x - 3 = y$ or $2x - y = 3$"	______	pair of lines
13	$\|x\| = 3$	______	pair of lines

Completing the table below will be a good review of your knowledge about quadratic polynomials.

	If the phrase	is	then k is:	
14	$x^2 + 5x + k$	a perfect square	______	$\frac{25}{4}$
15	$x^2 + kx + 3$	factorable over the integers	___ or ___	4 or -4
16	$9x^2 - 18x + k$	a perfect square	______	9
17	$x^2 - kx + 3$	a perfect square	___ or ___	$2\sqrt{3}$ or $-2\sqrt{3}$
18	$-x^2 - kx - 12$	$-(x + 12)(x + 1)$	______	13

Solve. Answers are on pages lxiv - lxv.

19. The perimeter of a rectangle is 94 feet, and its area is 496 square feet. Find its length. ________

20. An open box is constructed from a rectangular sheet of metal 8 inches longer than it is wide, as follows: Out of each corner, a square of side 2 inches is cut, and the sides are folded up. The volume of the resulting box is 256 cubic inches. What were the dimensions of the original sheet of metal? ________

21. The sum of a number and its reciprocal is 4. What is the number? ________

Write in standard form, give the vertex, and give the coordinates of the points, if any, where the parabola intersects the x-axis. Answers on page

22. $y = 3x(x - 3)$
23. $y = x^2 + 2x - 8$
24. $y = x^2 + 6$

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Chapter 24

FUNCTIONS

24-1. The Function Concept

In drawing the graph of an open sentence in two variables we were concerned with a certain set of ordered pairs--the members of the truth set of the open sentence.

Consider the sentence

$$y = 3x + 7.$$

The truth set of this sentence consists of

1 ordered ______ of real numbers.

Thus, if a is a real number, we may find the pair

2 of real numbers $(a, \_\_ + 7)$ for which this sentence is true.

For any real number a, there is one and only one

3 real number ______ associated with a so that | $3a + 7$

4 $(a, 3a + 7)$ belongs to the truth set of $y =$ ______. | $y = 3x + 7$

5 For example, the ordered pair $(1, \quad)$ belongs to the truth set of $y = 3x + 7$. | $(1, 10)$

6 Therefore, the real number ______ is associated with the number 1. | 10

We might say that we have a set of numbers, that is, the set of values of a; and a rule which associates with each member of this set exactly one real number; the corresponding value of $3a + 7$. This idea of associating with each member of a given set of numbers with exactly one member of a second set of numbers is of fundamental importance in mathematics and has wide applicability.

Let us consider a quite different situation which illustrates this idea. Here are some facts about the cost of first class postage.

Weight in Ounces	Total Cost in Cents
1 or less.	5
2 or less but more than 1.	10
3 or less but more than 2.	15
4 or less but more than 3.	20

Et cetera.

No parcels over 20 pounds (320 ounces) can be accepted for first class mail.

If you want to find out how much it costs to mail a certain first class package, you need to know its weight in ounces.

For example, a parcel which weighs $1\frac{3}{4}$ ounces will
7 cost ______ ¢ to mail. 10¢

And if the parcel weighs $4\frac{3}{4}$ ounces it will cost
8 ______ ¢, since the table indicates that the same pattern continues for parcels heavier than 4 ounces. 25¢

Suppose a parcel weighs 20 pounds and 15 ounces.
9 You ______ (can, can't) mail it. can't

10 20 pounds, of course, is ______ ounces. 320

We can determine that the cost of mailing a first class package weighing x ounces, provided x is a
11 real number and $0 < x \leq$ ______ . $0 < x \leq 320$

With every real number x such that $0 < x \leq 320$,
12 we may associate exactly ______ number, which is the cost of a first class package weighing x ounces. one number

Notice that every entry in the "Total Cost" column of
13 our table is a multiple of ______ . 5

The greatest possible cost of mailing a first class
14 package is 320×5 cents, or $ ______ . $16.00

Thus our information about first class rates furnishes a rule for associating with any member of the set of real numbers x, $0 < x \leq 320$, exactly one member of the set of multiples of 5, $\{5, 10, 15, \ldots, 1600\}$.

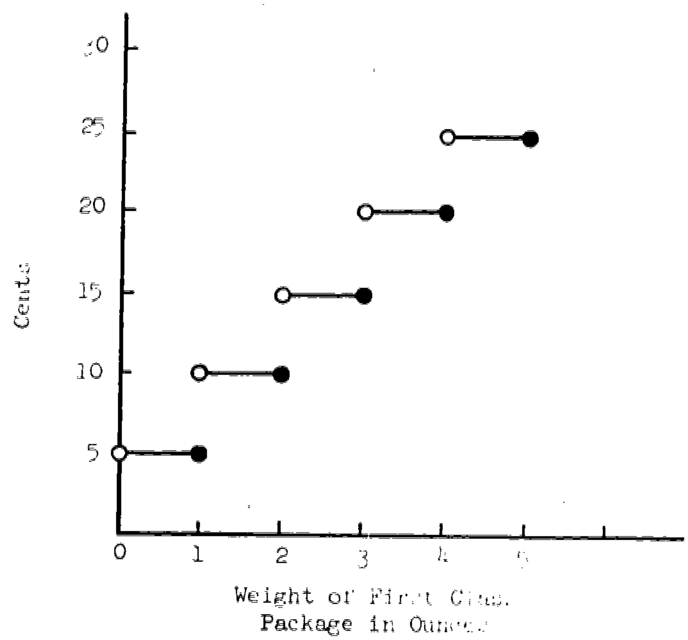
In the association which we have been considering:

15 The number _____ is associated with $3\frac{1}{2}$. 20

16 The number _____ is associated with 6. 30

17 The number _____ is associated with .02. 5

The information that we have been given could be represented by a graph. Let us draw a portion of such a graph.



Notice that this graph is different from those which we have seen before. Can you interpret this graph?

Which of the following points are on the graph?

18 (2,10) _____ (is, is not) is

19 (2,15) _____ is not

20 (.13,5) _____ is

([illegible])	________	is
(0,5)	________	is not

[illegible] a first class package costs [illegible]¢ to mail and if [illegible] is [illegible] then the number [illegible] is [illegible]:

[A] [illegible] [C] [illegible]

[B] [illegible] [D] [illegible]

The correct choice is [D]. If you made the wrong choice, study the graph carefully.

[illegible] that there is no simple "formula", or expression in [illegible] variable, [illegible] automatically tells us the cost in cents of mailing a first class package of a given weight.

The importance of this example about the cost of first class postage is:

> We have a set of numbers (the weight in ounces), and a rule which tells us how to associate with each number of this set a number (the cost in cents).

Let us examine some other examples.

Look at the table below.

Positive integer n	1	2	3	4	5	6	7	8	9	...
nth odd integer	1	3	5	7	[illegible]					

In the first row of the table we see the first nine

24 positive ______. — integers

With each number n in the first row, we associate

25 the nth ______ integer, as shown in the second row. — odd

26 Thus 7 is associated with ______, because 7 is the 4th odd integer. — 4

The numbers that should appear in the empty boxes in

27 the above table, in order, are: ____, ____, ____, ____. — 11,13,15,17

With each positive integer n there has been

28 associated the nth ______ integer. — odd

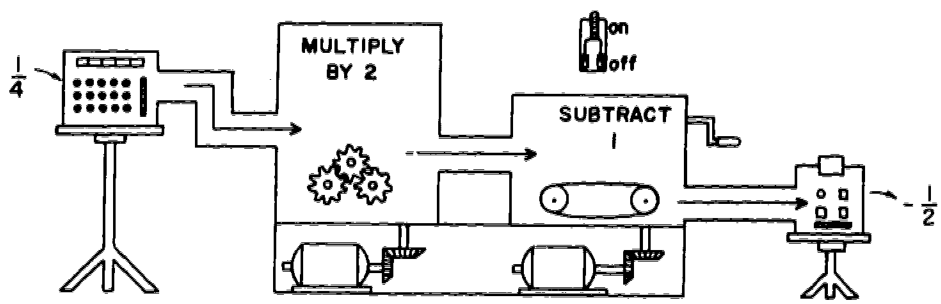
13 is in the set of positive integers (first row), and we can associate with it the 13th odd integer,
29 namely _____. 25

With 1000 we can associate the 1000th odd integer,
30 namely _____. 1999

[Have you noticed that the nth odd integer is $2n - 1$?]

Thus, we have a set, the set of positive integers, and a rule associating with each member of this set exactly one positive odd integer. In this case we have an association between the set of positive integers and the set of positive odd integers.

Here is a sketch of an imaginary computing machine. It accepts any positive real number.



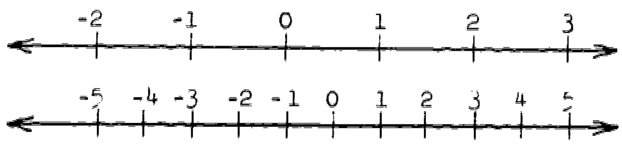
If the machine is "fed" the positive number 4, the
31 machine yields the number _____. 7

Complete the following:

	In	Out	
	In	Out	
32	1	_____	1
33	$\frac{1}{2}$	_____	0
34	10	_____	19
35	25	_____	49
36	.01	_____	-.98
	0	The machine jams. [0 is not a positive number.]	

It is important to notice that this machine provides us with a "rule" for associating with every member of the set of positive real numbers exactly one real number.

Examine the following two number lines, which are drawn using different units of measure.



To each point on the upper line, there corresponds exactly one point on the lower line.

	Point on Upper Line	Point on Lower Line	
	2	3	
37	1	____	1
38	$\frac{1}{2}$	____	0
39	10	____	19
40	0	____	-1
41	-2	____	-5

Notice that in each case, if n is the coordinate of the point on the upper line, then $2n - 1$ is the corresponding coordinate on the lower line. Our pair of lines operates somewhat as our "machine". In this case the association is from the set of all real numbers to the set of all real numbers.

Consider a line whose slope is 2 and whose y-intercept is (0,-1). For any point of this line the ordinate is found by first multiplying the abscissa by 2 and then subtracting 1. Do you see that this line provides a rule for associating with every real number (the abscissa) exactly one number (the ordinate)?

Suppose we are given the following verbal instructions: given any negative real number multiply it by 2 and then subtract 1. Do you see that this verbal instruction provides a rule for associating exactly one real number with every negative number?

Thus far, most of our illustrations have been associations which assign exactly one number to every member of a given set. Other associations may assign more than one number. Here are some examples of this type.

	"T[illegible] number d assign [illegible] such that the number pair (d,c) is a [illegible]."	
42	If [illegible] is [illegible], this rule assigns ______ (how many) [illegible].	two [3 and -3]
	"T[illegible] assign [illegible] that [illegible]."	
43	If [illegible] is [illegible], this rule of association assigns ______ (how many) [illegible].	two [3 and -3]
	"To each integer p assign a real number q such that q < p."	
44	If p is [illegible], this association assigns an infinite set of numbers to _____.	q

The particular associations that we wish to study are those that we have emphasized--those associations which assign exactly one number to each member of a given set.

Definition. Given a set of numbers and a rule which assigns to each number of this set exactly one number of a second set, the resulting association of numbers is called a function. The given set is called the domain of definition of the function, and the set of assigned numbers is called the range of the function.

It is very important to understand that two different rules give the same function if and only if they involve the same domain of definition and determine the same association of numbers.

Thus the following three rules give the same function:

"Given a positive real number, multiply it by 2, then subtract 1. Associate the number obtained with the given number."

"Given the abscissa of a point on the graph of $y = 2x - 1$ and $x > 0$, associate the ordinate of the given point with the abscissa of the given point."

"Feed a positive real number to the machine of Items 31 - 36. Associate the number turned out with the number fed in."

Item	Frame	Answer
45	A function involves a given set of numbers and a rule for associating with each member of this set with ______ (how many) real number(s).	one
46	The domain of definition of a function is the given ______.	set
47	In order to have a function, we must have a rule for associating exactly one real number with each element of the domain of definition of the ______.	function
48	When we have a function, exactly one number is assigned to each element of the ______ of ______.	domain of definition
	The set of assigned numbers is called the range of the function.	
49	The ______ of a function is the set of assigned numbers.	range
50	The range of a function is thus a ______ of numbers.	set
51	Two descriptions lead to the same function provided that we have the same ______ of definition, and both	domain
52	descriptions assign the same number to each member of the ______.	domain

For our purpose, we shall be interested in describing a function by an expression in one variable, since it allows us to use algebraic methods to study the function. On the other hand, it should be realized that a function need not be described by an expression in one variable. (Recall the example of the first class postage.) The graphical method of describing a function is also important since it enables us to visualize certain properties of a function.

Consider the following description:

Given any real number, square it, add 2 times it, then subtract 3.

53 Has a function been described? ______ (yes,no) **yes**

54 The domain is the set of ______ ______. **real numbers**

55 What number is assigned to 1? ______ **0**

56 What number is assigned to -3? ______ **0**

May the same number be assigned to two different

57 members of the domain? ______ **yes**

May one member of the domain have two different

58 numbers assigned to it? ______ **no**

The last response is "no". If the answer were "yes" we would not be dealing with a <u>function</u>.

If we use the description above, we can write an expression in one variable which tells which number to assign to x.

59 This expression in x is x^2 ______. **$x^2 + 2x - 3$**

Remember, if a rule assigns more than one number to <u>any</u> member of a domain then the rule does not describe a function. Further, to <u>every</u> member of a domain there must be some number assigned by the rule.

60 Which of the following describes a function having as domain the set of all real numbers?

[A] $\sqrt{2}x^2 - \frac{1}{\sqrt{3}}$ [C] $\sqrt{x}$

[B] $\frac{1}{x}$ [D] $\sqrt{|x|}$ and $-\sqrt{|x|}$

[B] does not represent a number if x is 0. [C] does not represent a number if $x < 0$. [D] assigns two values to every x except 0. [A] is the correct choice.

With each positive integer associate its remainder after division by 5.

61 The domain of this function is the set of ______ ______. **positive integers**

62	With [illegible] we associate the number ______.	2
63	With [illegible] we associate the number ______.	0
64	The range of this function is {0, ________}.	{0,1,2,3,4}

	To each positive real number assign the product of $\frac{1}{3}$ and 2 more than the number.	
65	The domain of this function is the set of ______ ______ numbers.	positive real
66	To 4 we assign $\frac{1}{3}(4 + 2)$ = ______.	2
67	To $\frac{1}{2}$ we assign the number ______.	$\frac{5}{6}$
	An expression that describes this function is:	
68	to each positive real number x assign $\frac{1}{3}$( ).	$\frac{1}{3}(x + 2)$
69	The range of the function ______ (does, does not) include the number $\frac{2}{3}$.	does not

	With each positive integer n, associate the nth prime.	
70	The domain is the set of ______ ______.	positive integers
71	The range is the set of ______.	primes
	With 1, we associate the prime 2.	
	With 2, we associate the prime 3.	
72	With 3, we associate the prime ____.	5

	Associate with each day of the year the number of days remaining in the (non-leap) year.	
73	Domain of definition: Set of all ______ ______ less than 366.	positive integers
	Tell what numbers you would insert in the boxes to complete the following table.	
74	(see table below)	364, 265, 1
75	Algebraic expression: With each positive integer n, less than 366, associate ______.	365 - n

n	1	100	364
days remaining on nth day			

Associate with each positive real number the length of the circumference of the circle having the number as the diameter.

76 Domain of definition: Set of _______ _______ numbers. **positive real**

Tell what numbers complete the table below by supplying the missing circumferences (in terms of π).

diameter, d	1	2	10
circumference	π		

77 **2π, 10π**

Algebraic expression: With each positive real number

78 d associate _______. **πd**

Assign to each real number x the number -1 if x is rational, and the number 1 if x is irrational.

79 The domain of definition: Set of _______ _______ numbers. **all real**

80 Range of the function: _______ **$\{-1,1\}$**

What numbers are assigned to each of the following?

	To	Assign	
	$-\pi$	1	
81	$-\frac{3}{2}$	____	**-1**
82	$-\sqrt{2}$	____	**1**
83	0	____	**-1**
84	$\frac{1}{2}$	____	**-1**
85	$\sqrt{7}$	____	**1**

We have seen many examples of functions. In each of them the domain of definition was stated. If the domain of definition is not stated we shall assume that it is the largest set of real numbers to which the rule defining the function can be applied.

For example, consider the function defined by $\frac{1}{x+2}$. In this case, we should understand that the domain of definition is the set of all real numbers except -2.

	For the expression:	The domain is the set of real numbers x, such that:	
	$\frac{x}{x-3}$	$x \neq 3$	
86	$1 - \frac{1}{x}$	$x \neq$ ____	$x \neq 0$
87	$\sqrt{2x+1}$	$x \geq$ ____	$x \geq -\frac{1}{2}$
88	$\frac{1}{x^2-4}$	____ and ____	$x \neq 2$, $x \neq -2$
89	$\sqrt{x^2-1}$	$\lvert x \rvert$ ____	$\lvert x \rvert \geq 1$

We often deal with functions that arise from physical problems. For any real number s, $s(5 - s)$ is a real number. Suppose, however, we are concerned with the area of a rectangle of perimeter 10 inches.

If we let s be the length of one side, in inches, then the other side of the rectangle will be
90 $5 -$ ____ inches. — $5 - s$

For each value of s, $s(5 - s)$ defines a number, the area of the rectangle, which may be associated with s.

In this case we would restrict the domain of s, so
91 that ____ > 0, since the length of a side must be positive. — $s > 0$

Likewise, since the total perimeter is 10, s must
92 be less than ____. — 5

Thus, the domain of s, for this problem is the set of
93 of values such that ____ $< s <$ ____. — $0 < s < 5$

Suppose we are thinking about triangles which have areas of 12 square inches.

94 (Remember that the area of a triangle is ____, when b is the base and h the altitude.) — $\frac{1}{2}bh$

This formula $(A = \frac{1}{2}bh)$ shows the relation between
95 the ____, the altitude and the area of a triangle. — base

	If we know the length of the base of a triangle, and we know that the area of the triangle is 12 square	
96	inches, we can find the length of the ______.	altitude
	Thus we can consider the function by which the base	
97	length is associated with the length of the ______.	altitude
	(Remember the area of the triangle is 12 square	
98	inches and the formula we are using is ______.)	$A = \frac{1}{2}bh$
	A suitable algebraic expression for this function is:	
99	With the number b, associate ______.	$\frac{24}{b}$
	The appropriate domain is the set of real numbers b	
100	such that ______.	$b > 0$

We have seen a variety of situations in which the idea of function appears. It would be very convenient to have a special symbolism to use in discussing functions. This symbolism is introduced in Section 24-2.

24-2. The Function Notation

It is convenient to use some symbols to express the fact that we are considering a function.

Although we have used letters as names for numbers, we will also use letters as names for functions.

If f is a given function, and x is a number in its domain of definition, then by f(x) we shall mean the number which f associates with x.

The symbol f(x) is read "f of x", and not "f times x".

1	f(2) is read "______ of 2".	f of 2
2	g(x) is read "____ ____ ____".	g of x

	We must be quite careful to observe that f(x) is a number.	
3	It is the ______ which the function f assigns to the number x.	number

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The number $f(x)$ is called the <u>value of</u> f <u>at</u> x.

4 $f(2)$ is the ______ of f at 2. — **value**

5 $g(3)$ is the value of ____ at ____. — **g at 3**

6 $h(\frac{1}{2})$ is the ______ of ______ at ______. — **value of h at $\frac{1}{2}$**

If f is a function and if x is in the domain of definition of f, then $f(x)$ is a number in the

7 <u>range</u> of ____. — **f**

Consider $g(-3)$.

8 This is a number in the ______ of g, provided that -3 is in the domain of definition of g. — **range**

Consider $h(0)$.

If 0 is in the domain of definition of h, then

9 $h(0)$ is a ______ in the range of h. — **number**

Suppose we desire to deal with the following function f: "To each real number x assign the real number $2x - 1$".

10 The domain of f is the set of all ______ ______. — **real numbers**

Let x be 3. We see that f assigns the number

11 ____ to 3. — **5**

We would write: $f(3) = 5$.

12 Similarly: $f(\frac{1}{4}) = 2(\frac{1}{4}) - 1 =$ ____. — **$-\frac{1}{2}$**

13 $f(0) =$ ____. — **-1**

14 $f(-\frac{1}{2}) =$ ____. — **-2**

15 $f(\frac{1}{2}) =$ ____. — **0**

"$f(x) = 2x - 1$ for any real number x" is a complete statement of the desired function.

[illegible] notation.

If $f(x) =$ [illegible] x, then

16 $f(-$ [illegible] $) =$ [illegible] $=$ ______. — $-\frac{11}{3}$

17 $f($ [illegible] $) =$ ______. — $\frac{5}{3}$

18 $f(0.$ [illegible] $) =$ ______. — -0.4

19 $f($ [illegible] $) =$ ______, [illegible] — $2a - 1$

If [illegible]

20 $f($ [illegible] $) =$ ______. — $2t - 1$

21 $2f($ [illegible] $) =$ ______. — $2(2t - 1)$, or $4t - 2$

22 If [illegible] is a real number, and the function f is the same as in Items 10 - 21, which of the following are true for any real number [illegible]?

K. $f(-$ [illegible] $) =$ [illegible]

S. $-f($ [illegible] $) =$ [illegible] $+$ [illegible]

T. $f($ [illegible] $-$ [illegible] $) =$ [illegible]

U. $f($ [illegible] $) -$ [illegible]

[A] K, T, and U are true.

[B] [illegible] true.

[C] K, S, and U are true.

[B] is the correct choice. If you were wrong--or not sure--complete Items 23 to 28. Otherwise, skip to Item 29.

$f(-b) =$ [illegible] $(-$ [illegible] $) -$ [illegible]

23 $=$ ______ — $-2b - 1$

However, since $f($ [illegible] $) =$ [illegible], we see that

$-f(b) = -(2b -$ [illegible] $)$

24 $=$ ______ — $-2b + 1$

Is it true for this function that $f(-b) = -f($ [illegible] $)$.

______ (yes, no) — no

$f(b - 1) = 2(b - 1) - 1$

26 $=$ ______ — $2b - 3$

480

[illegible] ________	2b - 2
Is it true for this function that [illegible] $\overline{\text{(yes, no)}}$	no

Let $F(x) = 2 - \frac{x}{2}$ for any real number x.	
Instead of f denoting a function we are letting ________ denote the function.	F
$F(-2) =$ ________.	3
$-F(2) =$ ________.	-1
$F(-\frac{1}{2}) =$ ________.	$\frac{9}{4}$
$F(3) = 2 -$ ________.	$\frac{1}{2}$
$F(0) =$ ________.	2
$F(-6) =$ ________.	5
$F(6) =$ ________.	-1
$F(t) =$ ________, for any real number t.	$2 - \frac{t}{2}$ or $\frac{4-t}{2}$
$F(\frac{t}{2}) =$ ________.	$2 - \frac{t}{4}$ or $\frac{8-t}{4}$
$F(2t) =$ ________.	2 - t
40 $F(\frac{1}{t}) =$ ________.	$2 - \frac{1}{2t}$ or $\frac{4t-1}{2t}$

Let $H(z) = z^2 - 1$ for $-3 < z < 3$.	
41 $H(2) = (2)^2 - 1 =$ ________.	3
42 $-H(2) =$ ________.	-3
43 $H(-1) =$ ________.	0
44 $H(-1) + 1 =$ ________.	1
45 Is $H(5)$ a real number? $\overline{\text{(yes, no)}}$	**no [5 is not an element of the domain of definition.]**
It is not surprising that we refer to $H(z)$ as a <u>quadratic</u> function.	

Let $g(y) = 3y - 6$.

46 $g(2) =$ ______.

0

$g(y) = 0$ is an open sentence. The truth set is the set of numbers a, for which the value of $g(a)$ is 0.

47 The truth set of $g(y) = 0$ is ______.

{2}

(Notice that we solved the equation $3y - 6 = 0$.)

48 The truth set of $g(y) = -9$ is ______.

{-1}

(Notice that we solved the equation $3y - 6 = -9$.)

49 The truth set of $g(y) > 0$ is the set of all real numbers ______ (less, greater) than 2.

greater

(Notice that we solved the inequality $3y - 6 > 0$.)

50 The truth set of $g(y) < 6$ is the set of all real numbers ______ (greater, less) than 4.

less

Given $F(x) = 2 - \frac{x}{2}$ for all real numbers x, find the truth set of:

51 $F(x) = -1$ ______

{6}

52 $F(x) < 0$ ______

Set of all numbers greater than 4.

53 $F(x) = -\frac{1}{2}$ ______

{5}

54 $F(x) = x$ ______

$\{\frac{4}{3}\}$

55 $F(x) \geq 2$ ______

Set of all numbers less than or equal to 0.

56 $F(x) \leq 1$ ______

Set of all numbers greater than or equal to 2.

It is pleasant when a simple algebraic expression can be used to define a function. But remember, we have defined a function whenever we assign--by any sort of rule--exactly one number to every element of the domain of definition.

Sometimes [illegible] pieces. For instance, a function h is defined by

$h(x) = x$, for each number x such that $x \geq 0$

$h(x) = -x$, for each number x such that $x < 0$.

This is a single rule, and defines one function, even though it involves two equations. To make this fact clear, we write

$$h(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

What is the domain of definition of the function h, and what is the range of h?

[A] The domain of definition of h is the set of real numbers. The range of h is the set of real numbers.

[B] The domain of definition of h is the set of real numbers. The range of h is the set of non-negative real numbers.

[C] The domain of definition of h is the set of non-negative real numbers. The range of h is the set of real numbers.

Notice that h assigns a number to any real number x. However, for any real number x we see that $h(x) \geq 0$. Thus [B] is the correct choice.

5? Which of the following is also a correct way of describing this function h?

R. $h(x) = |x|$

S. $h(x) = \sqrt{x^2}$

T. $h(x) = -x$

[A] R only

[B] R and S

[C] R and T

Note that if x is any real number $h(x)$ is non-negative. Hence, it is not true that $h(x) = -x$ for all numbers x in the domain of the function. [B] is the correct choice. Review the definitions of $|x|$ and $\sqrt{x^2}$ if you weren't sure.

	Here is another example of a function having the set of all real numbers as domain of definition. Consider $\begin{cases} g(x) = -1 \text{ for each real number } x \text{ such that } x < 0, \\ g(x) = 0 \text{ for } x = 0, \\ g(x) = 1 \text{ for each real number } x \text{ such that } x > 0. \end{cases}$ Again we may state this rule in our abbreviated notation, as follows:	
59 60	$g(x) = \begin{cases} -1, & x < 0 \\ 0, & \underline{x = \quad} \\ \_\_\_, & \_\_\_\_ \end{cases}$	$x = 0$ $1, \quad x > 0$
61	The domain of g is the set of ___ ___ ___.	all real numbers
62	The range of g is ___}.	{-1,0,1}
63	$g(-5) =$ ___.	-1
64	$g(0) =$ ___	0
65	$g(1007) =$ ___	1
66	$g(10,3[illegible]) =$ ___	1

	Compare the functions: $g(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$ and $h(t) = \begin{cases} -1, & t < 0 \\ 0, & t = 0 \\ 1, & t > 0 \end{cases}$	
67	Do $g(x)$ and $h(t)$ have the same domain? ___ (yes,no)	yes
68	Do $g(x)$ and $h(t)$ define the same rule of association? ___ (yes,no)	yes
69	For any real number a, is $g(a) = h(a)$? ___ (yes,no)	yes
70	Is g the same function as h? ___ (yes,no)	yes

	Consider $g(x) = \frac{\lvert x \rvert}{x}$.	
	The [illegible] of $g(x)$ is the set of all real numbers except ____.	0
	$g(-1) =$ ____	-1
	$g(1) =$ ____	1
	Is g the same function as [illegible] defined by:	
	$g(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$? ____ (yes, no)	no [The domains are different.]

Remember that [illegible] the domain of definition. In this [illegible] numbers for which the [illegible] defining the function [illegible]. [illegible] say simply "[illegible] $f(x) = \frac{1}{x}$," [illegible] that the domain is [illegible] except 0.

	Consider the two functions specified by: $f(x) = x - 1$; $F(x) = \frac{x^2 - 1}{x + 1}$	
75	$f(x)$ and $F(x)$ ____ (do, do not) represent the same function.	do not
76	They have ____ (the same, different) domain(s) of definition.	different
77	The domain of definition of ____ contains -1, while the domain of definition of F does not.	f

	Consider the two functions defined by: $f(x) = x^2 - 1$, $G(t) = \frac{t^4 - 1}{t^2 + 1}$	
78	$f(x)$ and $G(t)$ ____ (do, do not) define the same function.	do
79	The fact that different variables are used ____ (does, does not) make the functions different.	does not

	[illegible] (x) [illegible] $g(x)$ have the same domain of definition,	
39	[illegible] ________ ________, [illegible]	real numbers
41	For any real number a, $\frac{a^2 - 1}{[illegible]} =$ ________.	$a^2 - 1$
42	[illegible] (a) $\underset{(=, \neq)}{\_\_\_\_\_\_}$ $g(a)$ for all real numbers a.	$=$

24-[illegible]. Graphs of Functions

We have seen in [illegible] that one way to define a function is by means of a graph. Moreover, if a function f is defined in *any* way, we can represent it by a graph. *If a* is in the domain of f and if [illegible] is the [illegible] range associated with a, then the point $(a, [illegible])$ is on the graph of f. The graph of f consists of all such points.

1	If $f(x) = [illegible]x - 1$, $0 \leq x < 2$, then the graph of f is the graph of the ________ ________ of the compound open sentence "$y = [illegible]x - 1$ and $0 \leq x < 2$."	truth set
2	Likewise the graph of the function "$f(x) = x^2 - 1$ for all $x > 0$" is the graph of the truth set of the open sentence "________ and $x > 0$".	$y = x^2 - 1$

3. Draw the graph of the function

$$f(x) = [illegible]x - 1, \quad 0 \leq x < [illegible].$$

(Answer on page lxv .)

Notice in your graph that the point $(0,-1)$ is marked with a heavy dot, while the point $([illegible],[illegible])$ is circled. $([illegible],[illegible])$ is not a point on the graph, since the domain of definition of the function does not include [illegible].

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4. Consider the functions f and F given by:

$f(x) = 2x - 1, \quad 0 \le x \le 2.$

$F(x) = 2x - 1, \quad -2 < x < 2.$

Are the graphs of the two functions, f and F, the same?

[A] Yes [B] No

Even though both functions have a rule given by the polynomial $2x - 1$, their domains are different; therefore, their graphs cannot be the same. [B] is the correct choice.

5. Draw the graph of the function g defined by:

$$g(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

Compare your result with the one given on page lxv.

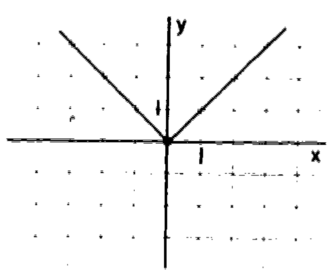
6. Draw the graph of $T(s) = 3s + 1, \quad -1 \le s \le 1.$
(Answer on page lxv.)

7. The function U is defined by

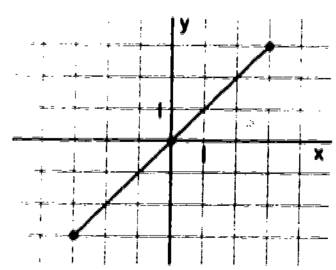
$$U(x) = \begin{cases} -x, & -3 \le x < 0 \\ x, & 0 < x \le 3 \end{cases}$$

Which of the following is the graph of U?

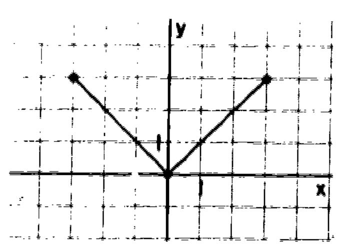
[A]



[B]



[C]



We can rule out [A] by noting that the domain of U is the set of real numbers between -3 and 3. We observe, too, that U(x) is non-negative for all values in the domain. [C] is the correct choice.

9 Draw the graph of $V(t) = t^2 - 1$, $-2 < t \leq 1$. Check with the answer on page lxv. Note the scale used.

(Hint: This should be familiar because you have graphed the equation $y = x^2 - 1$.)

The domain of definition of V is the set of all real numbers t such that ______. $-2 < t \leq 1$

10 The range is the set of all real numbers y such that ______. $-1 \leq y < 3$

11 Draw the graph of $h(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$

Check your result with that shown on page lxv.

12 The domain of definition of h ______ (is, is not) the set of all real numbers. is not [0 is excluded.]

13 The range of h is ______. $\{-1,1\}$

Consider the graph of $G(x) = \frac{|x|}{x}$.

The graph of this function is the same graph that you

14 drew in Item ______. (Examine page .) 11

Hence this function is the same as that given in

15 Item ______. 11

16. Draw the graph of

$$g(x) = \begin{cases} -1, & -5 \leq x < -1 \\ x, & -1 \leq x \leq 1 \\ x^2, & 1 < x \leq 2 \end{cases}$$

(Answer on page lxvi .)

We have observed that one way to define a function is to give its graph. We have also observed in many examples that the same function can often be described in more than one way.

Let us see if we can find another way to describe a function which is defined by the line segment extending from $(-2,-1)$ to $(4,5)$.

The line segment includes both endpoints.

Draw the line segment connecting $(-2,-1)$ and $(4,5)$. Check your drawing with the one in page lxvi .

We see that the domain of definition of the function represented by the graph is ______ $\leq x \leq$ ______. $-2 \leq x \leq 4$

The range is ______ $\leq y \leq$ ______. $-1 \leq y \leq 5$

By considering the graph you have drawn, you can see that when:

20 $x = -2$ $f(x) =$ ______ $f(x) = -1$

$x = 0$ $f(x) = 1$

21 $x = 1$ $f(x) =$ ______ $f(x) = 2$

$x = 3$ $f(x) = 4$

22 $x = 4$ $f(x) =$ ______ $f(x) = 5$

Since for each x, $f(x)$ is always $x + 1$, then the function is:

23 $f(x) =$ ______ for all numbers x such that $-2 \leq x \leq 4$. $f(x) = x + 1$

Find a rule for the definition of the function whose graph is the line segment connecting $(0,1)$ and $(6,4)$.

24 Draw the line segment connecting $(0,1)$ and $(6,4)$. (Check with the result, page lxvi.)

We note that the y-intercept number of the line

25 containing this segment is ______, and its slope 1

26 is ______. $\frac{1}{2}$

The function whose graph is the line segment connecting $(0,1)$ and $(6,4)$ can be described as follows:

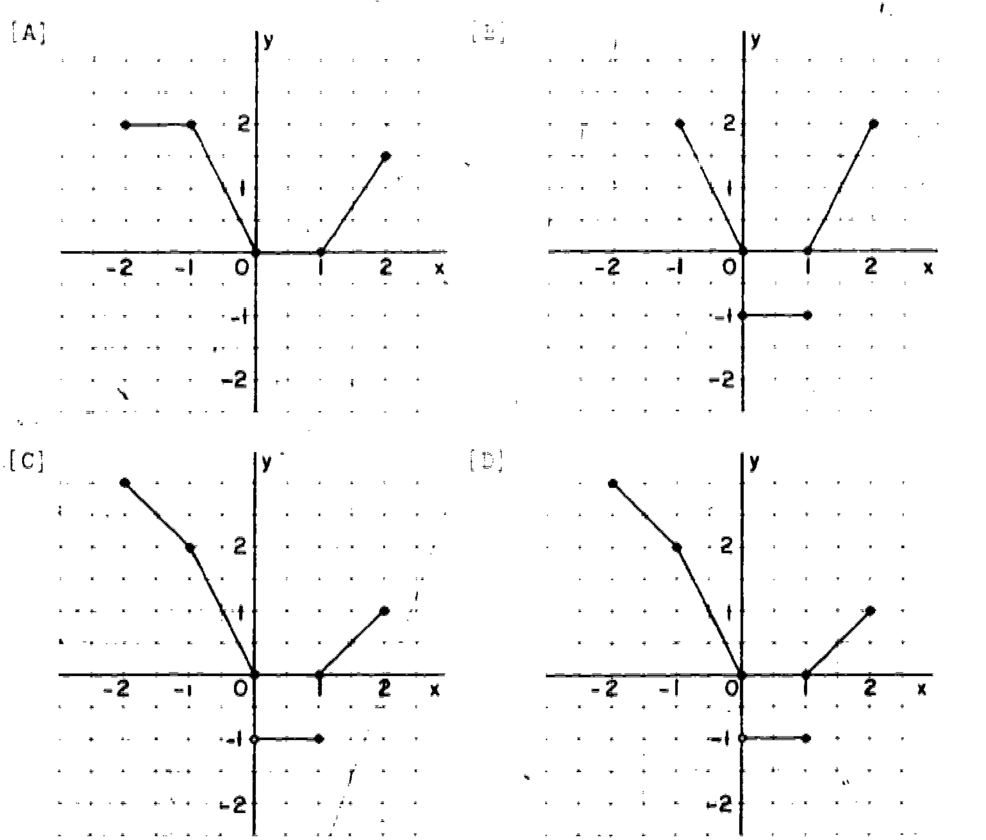
27 | $f(x) =$ ________, $0 \le x$ ________ | $f(x) = \frac{1}{2}x + 1$, $0 \le x \le 6$

28 Of the four graphs below, one is a graph of the function f which satisfies all of the following conditions over the domain of definition, $-2 \le x \le 2$:

$f(-1) = 2$ $\qquad$ $f(2) = 1$

$f(0) = 0$ $\qquad$ $f(x) < 0$ for $0 < x < 1$

$f(1) = 0$



The correct choice is [D]. If you made the wrong choice, complete Items 29 to 31. If not, omit these items.

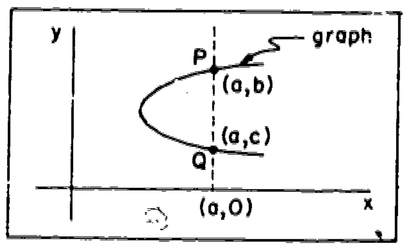
Graph [A] does not satisfy the condition "$f(x) < 0$ for $0 < x < 1$", since it contains the point $(\frac{1}{2}, \underline{\quad})$. — $(\frac{1}{2}, 0)$

The graph we are looking for contains the point $(\underline{2, \quad})$. [B], however, does not contain this point. — $(2,1)$

Graph [C] does not satisfy the condition $\underline{f(1) = \quad}$. — $f(1) = 0$.

Is every set of points the graph of some function? To answer this question, recall that if f is a function and a is a number in the domain of f, then there is exactly one number associated with a.

Consider the following situation:



The vertical line through $(a,0)$ intersects the graph in ______ (how many) points. — two

The abscissa of P is a, the ______ of Q is also a. — abscissa

The coordinates of P are $(a, \underline{\quad})$. — (a,b)

The coordinates of Q are $(a, \underline{\quad})$. — (a,c)

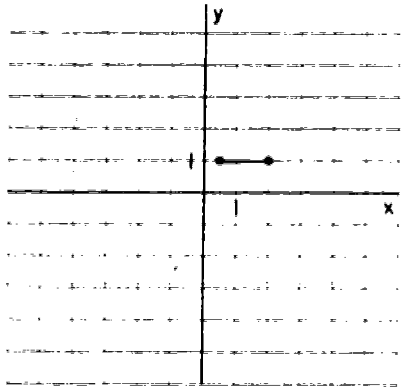
Since P and Q are different points, $b \neq c$.
Is the graph shown in the diagram the graph of a function? ______ (yes,no) — no

No, since the diagram shows two different numbers associated with the number a.

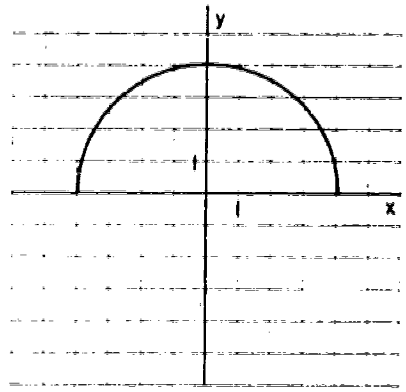
A vertical line intersects the graph of a function in at most ______ (how many) point(s). — one

38 Which of the following are graphs of functions?

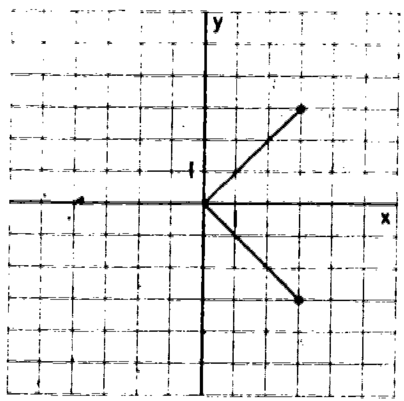
R.



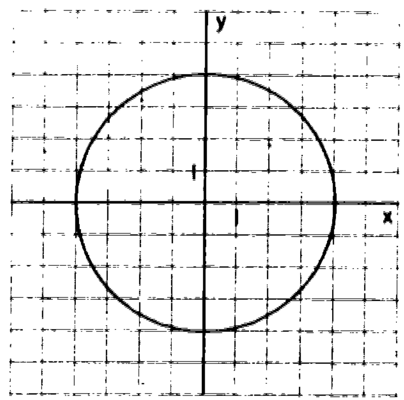
T.



S.



U.



[A] R only

[B] R and S only

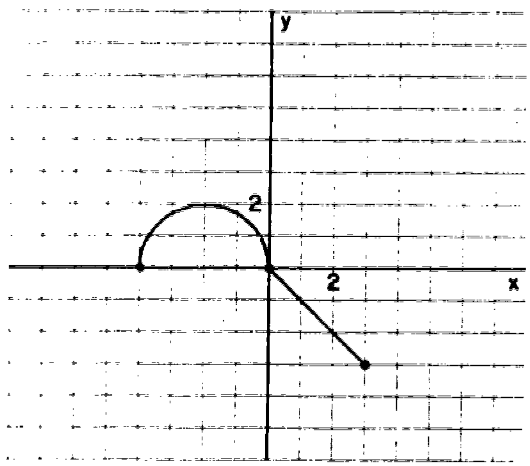
[C] R and T only

[D] R, S, and T

[E] All are graphs of functions.

A vertical line through (1,0) for [illegible] curve i[illegible] intersects the curve in U twice. [illegible]

The accompanying figure is the graph of the function f.



39 The domain of definition of the function f, represented by this graph is the set of all real numbers x such that _______. $-4 \leq x \leq 3$

40 The range of the function is the set of real numbers such that _______. $-3 \leq y \leq 2$

From the graph approximate:

41 $f(-3) \approx$ ______. ($\approx$ means approximately equal to.) 1.7

42 $f(0) =$ _______. 0

43 $f(2) =$ _______. -2

44 Draw the graph of $y^2 = x$, for $0 \leq x < 4$. (Hint: make a table of values.) Check your result with the graph on page lxvi .

45 The graph of $y^2 = x$, $0 \leq x < 4$ ________ (is, is not) the the graph of a function.

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24-4. Linear and Quadratic Functions

We have become familiar with the graphs of lines and parabolas. We can restate some of our conclusions about them using function terminology.

Consider $f(x) = x - 2$.

1 The graph of this function is a line with slope ______ **1**

2 and y-intercept (0, ____). **(0,-2)**

3 Since its graph is a ______, we refer to such a function as a linear function. **line**

4 $g(x) = 3x + 4$ is a ______ function. **linear**

5 $F(x) = -\frac{1}{2}x$ is a ______ ______. **linear function**

Any function f which can be expressed in the form $Ax + B$, where A and B are real numbers, is a linear function. If the domain of such a function is the set of all real numbers, then the graph is a line.

6 If the graph of f, where $f(x) = Ax + B$, is a line, then the slope of the line is ____, and the y-intercept is (______). **A** **(0,B)**

7 For the function $f(x) = 3$,

[A] the domain is {3}

[B] the range is {3}

The domain is the set of all real numbers.
[B] is correct.

8 Is the vertical line $x = 2$ the graph of a function?

[A] Yes

[B] No

Remember, we have a function when we have a rule associating with each number in the domain a single number in the range. A vertical line would indicate an association of all the real numbers with a single number (in this case, with 2). You should have chosen [B], since the line $x = 2$ is not the graph of a function.

We sometimes say "The perimeter of a square is a function of the length of a side."

This is consistent with our idea of function, since to every value of the length there corresponds exactly

9 one value of the _______.

10 This function can be expressed: $f(x) =$ _____.

11 It is a _______ function.

In this case we may say "the perimeter varies directly as the side." This means: the perimeter is a linear function of the side.

[redacted]

The cost in cents of some pencils, each costing 3¢,

12 is said to be a _______ _______ of the number of pencils. — **linear function**

13 The cost can be expressed by: $f(x) =$ _____. — **3x**

14 The cost varies _______ as the number of pencils. — **varies directly**

An interesting type of function is illustrated by the following example. Suppose a man travels 1 mile. Let x be his rate in miles per hour, and let y be his time in hours. We are led to the open sentence

$$y = \frac{1}{x}, \quad \text{where } x > 0.$$

(We needed to recall that rate × time = distance.)

In this case, we may say that the time in hours is a

15 function of the _______ in miles per hour. — **rate**

Since the function in question may be represented as

16 $f(x) =$ _______, we sometimes say that the time varies inversely as the rate. — **$f(x) = \frac{1}{x}$**

17 More generally, we say that one quantity varies _______ as another when the relevant function has the form $f(x) = \frac{k}{x}$. — **varies inversely**

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18 Complete the table of values for the open sentence $y = \frac{1}{x}$.

x	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3	4	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	-3	-4
y												

See answer below.

x	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3	4	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	-3	-4
y	3	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$

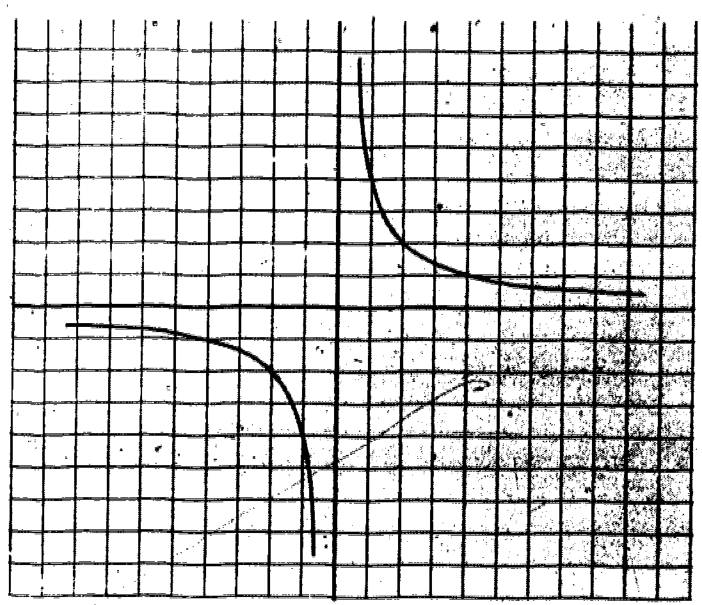
19 The domain of the function $f(x) = \frac{1}{x}$ is

[A] the set of all real numbers.

[B] the set of all real numbers except 0.

[B] is correct.

20 Draw the graph of the function $f(x) = \frac{1}{x}$. (See answer below.)



The graph of the function $\frac{1}{x}$ is called a <u>hyperbola</u>.

We have seen that if f is a linear function it can be described by

$$f(x) = Ax + B, \text{ where } A \text{ and } B \text{ are real numbers.}$$

It is natural to define a <u>quadratic</u> function as one which is expressible in terms of a quadratic polynomial,

$$f(x) = ax^2 + bx + c, \text{ where } a, b, \text{ and } c \text{ are real numbers and } a \neq 0.$$

	If $f(x) = ax^2 + bx + c$, where $a \neq 0$, and if the domain of x is the set of all real numbers, then	
21	the graph of the function is a ______.	**parabola**
	We recall that the parabola open upward if and only	
22	if a ______.	**$a > 0$**
23	If the parabola opens upward, then the ______ is the lowest point on the curve.	**vertex**
	If the parabola opens downward, the vertex is the highest point of the curve.	

	Consider the function $f(x) = 9x - x^2$.	
24	This is a ______ function, and its graph is a parabola	**quadratic**
25	that opens ______ (upward, downward).	**downward**
	To find the vertex, we may proceed as follows:	
	$9x - x^2 = -(x^2 - 9x)$	
26	$= -(x^2 - 9x + \_\_\_) + \frac{81}{4}$	**$\frac{81}{4}$, or $(\frac{9}{2})^2$**
27	$= -(x \qquad)^2 + \frac{81}{4}$	**$-(x - \frac{9}{2})^2$**
	$(x - \frac{9}{2})^2$ is non-negative for all values of x, and	
28	is 0 only when x has the value _____.	**$\frac{9}{2}$**
	The vertex of the parabola is $(\frac{9}{2}, \frac{81}{4})$.	
29	$(\frac{9}{2}, \frac{81}{4})$ is the ______ (highest, lowest) point on the curve.	**highest**

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Let us continue to consider the function $f(x) = 9x - x^2$. Since no specification as to domain is made, we take the domain to be the set of all real numbers.

The range of this function is the set of all real

30 numbers which are not greater than ____. — $\frac{81}{4}$

We might call $\frac{81}{4}$ the <u>maximum</u> value of the function.

31 $f(\quad) = \frac{81}{4}$ — $f(\frac{9}{2})$

Suppose we want to find two numbers whose sum is 9 and whose product is as large as possible.

If one number is x, then the other number is

32 $9 -$ ____, and their product is — $9 - x$

33 ____. — $x(9-x)$, or $9x-x^2$

For each value of x there is exactly one value of the product.

34 Hence the product is a ____ of x. — function

This function can be expressed as $f(x) = 9x - x^2$

35 since $x(9 - x) = 9x$ ____. — $9x - x^2$

We are asked to find the value of x for which the product is largest.

From Items 23 to 29, we know that the maximum value

36 of the product is ____. — $\frac{81}{4}$

The value of the function is $\frac{81}{4}$ when the value of

37 x is ____. — $\frac{9}{2}$

The two numbers whose sum is 9 and whose product

38 is as large as possible are ____, ____. — $\frac{9}{2}$, $\frac{9}{2}$

Suppose you wish to find two numbers whose sum is 12 and whose product is as large as possible. You might guess, from your result in the preceding items, that

39 the numbers are ___, ___, and that the product is — 6, 6

40 ____. — 36

As an exercise you can check for yourself, verifying this guess.

Consider two numbers whose <u>difference</u> is 12.

If x is one number, the product of the two numbers
41 is a ______ of x which can be expressed as — function
42 $f(x) = x(\_\_\_\_\_)$. — $x(x + 12)$

This is a quadratic function for which the graph is
43 a parabola that opens ______. — upward

44 Is there a <u>maximum</u> value for this function? ______ — no

However, there is a <u>minimum</u> value.

45 It is _____, since the vertex of the parabola is — -36
46 (______). — (-6,-36)

A boat manufacturer finds that his cost per boat in dollars is related to the number of boats manufactured each day by the formula:

$$c = n^2 - 10n + 175.$$

47 If he makes 1 boat his cost per boat is ______. — \$166

The number of boats he should manufacture each day
48 so that his cost per boat is smallest is ______. — 5

49 The cost per boat for this number of boats is ______. — \$150

Section 12-4

*42. Theorem 12-4c. For positive integers a, b, and c, if a is a factor of b, and a is a factor of (b + c), then a is a factor of c.

Proof:

There exists an integer p such that $b + c = ap$.	a is a factor of $b + c$.
There exists an integer q such that $b = aq$.	a is a factor of b.
$ap = aq + c$	Since $ap = b + c$ and $b = aq$.
$ap + (-aq) = c$	Addition property of equality.
$a\big(p + (-q)\big) = c$	Distributive property.
$p + (-q)$ is an integer.	The set of integers is closed under addition.
Hence, a is a factor of c.	Definition of factor.

Section 13-1

To show that $\frac{a}{-b} = -\frac{a}{b}$:

$\frac{a}{-b} = a(\frac{1}{-b})$, definition of division

$= a(-\frac{1}{b})$, theorem that $\frac{1}{-b} = -\frac{1}{b}$

$= -(a \cdot \frac{1}{b})$, $x(-y) = -(xy)$

$= -\frac{a}{b}$, definition of division

43. $\frac{2}{3}$, $x \neq 2$

44. $\frac{2}{3}$, $x \neq 2$

45. $\frac{y(x + 1)}{x + 1} = y$, $x \neq -1$

46. $x + 2$

47. $\frac{x + 2}{4}$

48. $\frac{(n + 3)(n + 2)}{6}$

49. 1, $n \neq -3$

50. 12a, $a \neq 0$

51. $t - 2$

52. 6, $x \neq 3$

53. y^2, $x \neq 1$, $x \neq -1$

54. $-\frac{3}{5}$, $a \neq 5$

Section 13-2

34. $\frac{(x + 3)(x - 3)}{x^2}$, $x \neq 0$

35. $\frac{7(2x - 7)}{2(7x - 2)}$, $x \neq 0$, $x \neq \frac{2}{7}$

36. $$\frac{2x - 7}{7x - 2} \cdot \frac{6}{21 - 6x} = \frac{(2x - 7)2 \cdot 3}{(7x - 2)(-3)(2x - 7)}$$

$$= -\frac{2}{7x - 2} \text{ or } \frac{2}{2 - 7x}, \quad x \neq \frac{2}{7}, \quad x \neq \frac{7}{2}$$

37. $\frac{2x - 7}{21} \cdot \frac{48y}{-2(2x - 7)} = -\frac{8y}{7}$, $x \neq \frac{7}{2}$, $y \neq 0$

38. $$\frac{45x(a + 3)}{343yz} \cdot \frac{14z(a + 9)}{15(a + 3)} = \frac{45 \cdot 14xz(a + 9)(a + 3)}{343 \cdot 15yz(a + 3)}$$

$$= \frac{3 \cdot 15 \cdot 2 \cdot 7xz(a + 9)(a + 3)}{15 \cdot 49 \cdot 7yz(a + 3)}$$

$$= \frac{6x(a + 9)}{49y}, \quad a \neq -3, \quad a \neq -9, \quad y \neq 0, \quad z \neq 0$$

Section 13-3

Proof that $\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}$.

$\frac{a}{c} + \frac{b}{c} = a(\frac{1}{c}) + b(\frac{1}{c})$	definition of division
$= (a + b)(\frac{1}{c})$	distributive property
$= \frac{a + b}{c}$	definition of division

75. $3|w| + 8 = \frac{1}{2}|w| + \frac{41}{2}$ (multiply by 2)

$6|w| + 16 = |w| + 41$

$5|w| = 25$

$|w| = 5$

$w = 5$ or $w = -5$ Truth set: $\{5, -5\}$

76. $-\frac{3}{7} + |x - 3| < \frac{22}{14}$ (multiply by 14)

$-6 + 14|x - 3| < 22$

$14|x - 3| < 28$

$|x - 3| < 2$ which may be interpreted as "the distance between x and 3 is less than 2." Thus the truth set is the set of all real numbers between 1 and 5.

77. If one of the numbers is n, the other is $\frac{3}{5}n$.

$n + \frac{3}{5}n = 240$

$5n + 3n = 1200$

$8n = 1200$

$n = 150$ Truth set: $\{150\}$ One number is 150, the other is 90.

78. If x is the amount by which the numerator is increased, then

$\frac{4 + x}{7} = \frac{27}{21}$

$3(4 + x) = 27$

$12 + 3x = 27$

$3x = 15$

$x = 5$ Truth set: $\{5\}$ The numerator was increased by 5.

79. If the father is x years old, Joe is $\frac{x}{3}$ years old.

$\frac{x}{3} + 12 = \frac{1}{2}(x + 12)$ (Multiply each side by 6.)

$2x + 72 = 3x + 36$

$36 = x$ Truth set: $\{36\}$. Joe is 12 years old, his father is 36 years old.

80. If x is the larger integer, then $(7 - x)$ is the smaller.

$x - (7 - x) = 3$

$x - 7 + x = 3$

$2x = 10$

$x = 5$ Truth set: $\{5\}$. The integers are 5 and 2. The reciprocal of the smaller integer decreased by the reciprocal of the larger is $\frac{1}{2} - \frac{1}{5} = \frac{5}{10} - \frac{2}{10} = \frac{3}{10}$

81. If x is the number of defective radios,

then $x = \frac{1}{20} \cdot 800$

$x = 40$ Truth set: $\{40\}$

The ratio of defective to non-defective radios is $\frac{40}{800 - 40} = \frac{40}{760} = \frac{1}{19}$

Perhaps you noticed that the number 800 is unnecessary information. If we suppose there were r radios in the shipment, then $\frac{1}{20}r$ were defective, and $\frac{19}{20}r$ were not defective.

$$\frac{\frac{1}{20}r}{\frac{19}{20}r} = \frac{1}{19}$$

Therefore the required ratio is $\frac{1}{19}$

86. $\frac{d}{7} + \frac{d}{8} = 1$

$8d + 7d = 56$

$15d = 56$

$d = \frac{56}{15}$ Truth set: $\{\frac{56}{15}\}$

d represents the number of days it requires Joe and Bob to paint the house.

87. Together, Joe and Bob will paint $(\frac{1}{7} + \frac{1}{8})$ or $\frac{15}{56}$ of the house in 1 day. Note that we could also reason as follows: If it takes $\frac{56}{15}$ days to paint the house, the fraction they paint in one day is

$$\frac{1}{\frac{56}{15}} = \frac{15}{56}.$$

Section 13-4

1. (a) $\frac{1}{12}$

(b) $-\frac{14}{105} = -\frac{2}{15}$

(c) $\frac{m - 18}{36}$

(d) $\frac{29a}{2\cdot2\cdot3\cdot11} = \frac{29a}{132}$

(e) $\frac{15}{14}$, $m \neq 0$

(f) $\frac{5\cdot24}{(\frac{1}{8} + \frac{1}{12})\cdot24} = \frac{5\cdot24}{3+2} = 24$

(g) $\frac{(5 - \frac{1}{a})\cdot5a}{(3a - \frac{3}{5})\cdot5a} = \frac{25a - 5}{15a^2 - 3a} = \frac{5(5a - 1)}{3a(5a - 1)} = \frac{5}{3a}$, $a \neq 0$, $a \neq \frac{1}{5}$

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(h) $\dfrac{(a + 7)(-7)}{(2a - 5)(3)(a + 7)} = -\dfrac{7}{3(2a - 5)}$, $a \neq -7$, $a \neq \frac{5}{2}$

(i) $\dfrac{7\cdot3(2x + 5)}{2(x - 2)\cdot3(2x + 5)} + \dfrac{(-5)\cdot2(x - 2)}{3(2x + 5)\cdot2(x - 2)} = \dfrac{42x + 105 - 10x + 20}{6(x - 2)(2x + 5)}$

$$= \frac{32x + 125}{6(x - 2)(2x + 5)}$$

$x \neq 2$, $x \neq -\frac{5}{2}$

2. (139)

3. If Kevin rides d miles into the hills, then he will ride d miles back. The number of hours riding into the hills is $\frac{d}{12}$ and the number of hours riding back is $\frac{d}{8}$.

$$\frac{d}{12} + \frac{d}{8} = 5$$

The distance is 24 miles.

Section 14-3

* Proof: If n is a negative integer and $a \neq 0$, then $a^{-n} = \frac{1}{a^n}$.

In words, the statement you want to prove is:

a^{-n} is the reciprocal of a^n.

Since n is negative,

$a^n = \frac{1}{a^{-n}}$ by definition ($-n$ is positive)

$a^{-n} \cdot a^n = a^{-n} \cdot \frac{1}{a^{-n}}$

$= 1$, by definiti... of reciprocal

Hence, a^{-n} is the reciprocal of a^n and therefore $a^{-n} = \frac{1}{a^n}$.

Likewise, $\frac{1}{a^{-n}} = a^n$.

Section 14-5

1. (a) $\frac{31}{26}$

(b) The least common denominator is $2\cdot3\cdot5^2$ or 150

$$\frac{3w}{5^2}\cdot\frac{6}{2\cdot3} - \frac{w}{2\cdot3\cdot5}\cdot\frac{5}{5} + \frac{2w}{3\cdot5}\cdot\frac{10}{2\cdot5} = \frac{18w - 5w + 20w}{2\cdot3\cdot5^2}$$

$$= \frac{33w}{150}$$

$$= \frac{11w}{50}$$

(c) The least common denominator is $2^2 \cdot 3a^2$.

$$\frac{2}{3a^2}\cdot\frac{4}{2^2} - \frac{5}{2^2\cdot 3a}\cdot\frac{a}{a} + \frac{1}{2^2}\cdot\frac{3a^2}{3a^2} = \frac{8 - 5a + 3a^2}{12a^2}$$

(d) The least common denominator is $2^4\cdot 3x^4y^3$.

$$\frac{5}{2^2\cdot 3x^4}\cdot\frac{4y^3}{2^2y^3} - \frac{1}{2\cdot 3x^3y}\cdot\frac{8xy^2}{2^3xy^2} + \frac{3}{2^4y^3}\cdot\frac{3x^4}{3x^4} = \frac{20y^3 - 8xy^2 + 9x^4}{48x^4y^3}$$

2. (a) $\frac{2\cdot 3\cdot 17}{2^2\cdot 3\cdot 5} = \frac{17}{10}$

(b) $\frac{2\cdot 5^2\cdot 7\cdot a^2}{2\cdot 5^2\cdot 3^2a} = \frac{7a}{9}$

(c) $\frac{3\cdot 17x^3y}{5\cdot 17xy} = \frac{3x^2}{5}$

3. 130 is divisible by 2.

131 is not divisible by 2, 3, 5, 7, nor 11. Since the next prime number after 11 is 13, and $13^2 = 169$, we see that 131 is prime.

4. If x is the integer, then $x + 1$ is its successor.

An appropriate open sentence is

$$4x = 2(x + 1) + 10$$
$$4x = 2x + 2 + 10$$
$$2x = 12$$
$$x = 6$$

The integer is 6.

5. (a) $2x - 99 = 87$

$$2x = 186$$
$$x = 93$$

The solution of the equation is 93, but 93 is not in the domain since 3 is a factor of 93. The truth set is empty.

(b) $12(\frac{x}{3} + \frac{5}{12}) = 12(12 + \frac{1}{4}x)$

$$4x + 5 = 144 + 3x$$
$$x = 139$$

Since 139 is prime, the truth set is {139}.

(c) If there is a prime number x for which

$$3x^2 < 123$$

then $x^2 < 41$

$x < 7$ where x is a prime number.

The primes less than 7 are 2, 3, and 5.
The left member is $3(2)^2 = 12$ when x is 2. $12 < 123$
The left member is $3(3)^2 = 27$ when x is 3. $27 < 123$
The left member is $3(5)^2 = 75$ when x is 5. $75 < 123$
Thus the truth set is $\{2, 3, 5\}$.

(d) If there is a prime number x such that

$$|x - 10| < 3$$

then $7 < x < 13$

Thus the truth set is $\{11\}$.

6. (a) $6a^3$ (d) 2
   (b) $\frac{y}{4x}$ (e) $27a^3$
   (c) 1 (f) 10^4

7. (a) $a^3 + a^2$
   (b) $x^3y^2 + xy^5$
   (c) $6x^3 + 3x^2$
   (d) $mn^2 - m^2n$
   (e) $a^2(a + b) + b^2(a + b) = a^3 + a^2b + ab^2 + b^3$
   (f) $x + x^2$
   (g) $a(a^{-1} + b^{-1}) + b(a^{-1} + b^{-1}) = a^0 + ab^{-1} + a^{-1}b + b^0$

$$= 1 + \frac{a}{b} + \frac{b}{a} + 1$$

$$= 2 + \frac{a}{b} + \frac{b}{a}$$

8. (a) either Examples $3^2 = 9$, $4^2 = 16$
   (b) either Examples $2^3 = 8$, $3^3 = 27$
   (c) even
   (d) odd
   (e) even even × even = ? Try some numbers.

(f) odd   odd × odd = ?

(g) even   2 is a factor.

(h) odd   One less than an even number is an odd number.

(i) odd   since 2 divides 2^{10} but does not divide 3^{10}

(j) even   since 2 divides both 2^{10} and 6^{10}

9. (a), (c), (e), (f), (g), (h), and (k) are non-negative.

10. If x is the length of the side of the smaller square then $x + 1$ is the length of the side of the larger square.

$$(x + 1)^2 - x^2 = 27$$
$$x^2 + 2x + 1 - x^2 = 27$$
$$2x = 26$$
$$x = 13$$

The length of the side of the smaller square is 13 units.

Check: Area of smaller square is $13^2 = 169$.
Area of larger square is $14^2 = 196$.
They differ by 27.

11. If x is the number of nickels, then $41 - x$ is the number of dimes.

$$5x + 10(41 - x) = 335$$
$$5x + 410 - 10x = 335$$
$$-5x = -75$$
$$x = 15$$

Thus, Bill has 15 nickels.

Check: 15 nickels are worth \$.75.
41 - 15 or 26 dimes are worth \$2.60.
\$2.60 and \$.75 is \$3.35.

The information about his saving for 27 days is unnecessary. The requirement that there are more dimes than nickels is satisfied since Bill has 26 dimes and 15 nickels.

12. If x is the number of gallons of mixture removed, then the number of gallons of salt in the original mixture minus the number of gallons of salt removed equals the number of gallons of salt in the final mixture.

$.15(100)$ is the number of gallons of salt in the original mixture.
$.15x$ is the number of gallons of salt in the mixture removed.
$.10(100)$ is the number of gallons of salt in the final mixture.

An open sentence is:

$$\begin{aligned} .15(100) - .15x &= .10(100) \\ 15(100) - 15x &= 10(100) \\ 500 &= 15x \\ 33\tfrac{1}{3} &= x \end{aligned}$$

The number of gallons of mixture removed is $33\frac{1}{3}$.

Check: The ratio of the number of gallons of salt in the original mixture minus the number of gallons of salt removed to the number of gallons of final mixture should equal 10%.

$$\begin{aligned} \frac{15 - .15(33\frac{1}{3})}{100 - 33\frac{1}{3} + 33\frac{1}{3}} &= \frac{15 - 5}{100} \\ &= \frac{10}{100} \\ &= .10 \text{ or } 10\% \end{aligned}$$

13. If r is the number of miles the train goes in 1 hour, then $8r$ is the number of miles the train goes in 8 hours. $10r$ is the number of miles the jet goes in 1 hour. An open sentence is:

$$\begin{aligned} 10r &= 8r + 120 \\ 2r &= 120 \\ r &= 60 \end{aligned}$$

The rate of the train is 60 miles per hour. The rate of the jet is 600 miles per hour.

Check: In one hour the jet travels 600 miles.
In eight hours the train travels $8(60)$ or 480 miles.
600 is the 120 greater than 480.

14. If r is the rate in m.p.h. of one train, then $\frac{2}{3}r$ is the rate in m.p.h. of the second. When they meet the number of miles traveled by the second train equals 320. Since they traveled for $3\frac{1}{5}$ or $\frac{16}{5}$ hours, an open sentence is:

$$\begin{aligned} \frac{16}{5}r + \frac{16}{5}\left(\frac{2}{3}r\right) &= 320 \\ 15 \cdot \frac{16}{5}r + 15 \cdot \frac{16}{5}\left(\frac{2}{3}r\right) &= 15 \cdot 320 \\ 48r + 32r &= 4800 \\ 80r &= 4800 \\ r &= 60 \end{aligned}$$

The rate of the faster train is 60 m.p.h.
The rate of the slower train is $\frac{2}{3}(60)$ or 40 m.p.h.

Check: $$\begin{aligned} 40(3\tfrac{1}{5}) + 60(3\tfrac{1}{5}) &= 128 + 192 \\ &= 320 \end{aligned}$$

15. If n is the number of lbs. of the \$1.00 candy, then
$40 - n$ is the number of lbs. of the \$1.40 candy.
$n(100)$ is the penny value of the \$1.00 candy in the mixture.
$(40 - n)(140)$ is the penny value of the \$1.40 candy in the mixture.
$40(110)$ is the penny value of the \$1.10 mixture.

An open sentence is:

$$\begin{aligned} n(100) + (40 - n)140 &= 40(110) \\ 100n + 5600 - 140n &= 4400 \\ 1200 &= 40n \\ 30 &= n \end{aligned}$$

The number of pounds of \$1.00 candy to be used is 30.
The number of pounds of \$1.40 candy to be used is 10.

Check: 40 lbs. of the final mixture at \$1.10 per lb. will sell for \$44.

30 lbs. at \$1.00 is \$30.
10 lbs. at \$1.40 is \$14.

\$14 and \$30 is \$44.

Section 15-2

*56. To Prove: $\frac{1}{2}\sqrt{2} - 1$ is irrational.

Proof: (by contradiction)

Assume $\frac{1}{2}\sqrt{2} - 1$ is rational.

Then, since 2 is rational,

$2(\frac{1}{2}\sqrt{2} - 1)$, or $\sqrt{2} - 2$, is also rational,

because the set of rational numbers is closed under multiplication. Then, also, $(\sqrt{2} - 2) + (2)$, or $\sqrt{2}$, is rational since the set of rational numbers is closed under addition.

But, $\sqrt{2}$ is not rational (Theorem 15-2). Hence, our assumption is contradicted, and we conclude that $\frac{1}{2}\sqrt{2} - 1$ is irrational.

*57. To Prove: $\sqrt{3}$ is irrational.

Proof: (by contradiction)

Assume $\sqrt{3}$ is rational; that is, that there are positive integers a, b, having no common factor, such that

$$\frac{a}{b} = \sqrt{3}$$

$$\left(\frac{a}{b}\right)^2 = 3$$

Then

$$\frac{a^2}{b^2} = 3$$

$$a^2 = 3b^2$$

Since a^2 is a multiple of 3, then a is also a multiple of 3.

Hence, there is an integer c such that $a = 3c$. Therefore

$$(3c)^2 = 3b^2$$
$$9c^2 = 3b^2$$
$$3c^2 = b^2$$

Since b^2 is a multiple of 3, then b is also a multiple of 3. But, for a and b to have a common factor, 3, is a contradiction of our assumption. Hence $\sqrt{3}$ is irrational.

Section 15-[illegible]

50. $4(11) = 44$
51. $(2\sqrt{[illegible]})(3\sqrt{[illegible]}) = [illegible]\sqrt{[illegible]}$
52. $([illegible]\sqrt{2})([illegible]\sqrt{2}\cdot\sqrt{7}) = [illegible]\sqrt{7}$
53. $2\sqrt{77}$
54. $([illegible]\sqrt{[illegible]}\cdot\sqrt{[illegible]})(2\sqrt{[illegible]}\cdot\sqrt{[illegible]}) = [illegible]4\sqrt{6}$
55. $\sqrt{25} = 5$
56. $\sqrt{9}\cdot\sqrt{16} = 3\cdot 4 = 12$
57. $3 + 4 = 7$
58. $3 - 4 = -1$
59. $\sqrt{4}\cdot\sqrt{229} = 2\sqrt{229}$
60. $\sqrt{34}$

110. $\sqrt{2x}$, where x is non-negative

111. $x\sqrt{3x}$, where x is non-negative

112. $3|n|\sqrt{5}$

113. $60\sqrt{6x}$, where x is non-negative

114. $5y^3\sqrt{3y}$, where y is non-negative

115. $4a^2\sqrt{2}$

116. $25|x|$

117. $|x^3|\sqrt{5}$

118. x^2

119. $1000\sqrt{3x}$, where x is non-negative

Section 15-4

22. $\frac{\sqrt{3}}{5}$

23. $\frac{7}{|a|}$ and $a \neq 0$

24. $\sqrt{\frac{4y}{9y^3}} = \sqrt{\frac{4}{9y^2}}$
$= \frac{2}{3|y|}$ and $y \neq 0$

25. $\frac{\sqrt{2}}{3|a|}$ and $a \neq 0$

26. $\frac{2}{3}$

27. $\sqrt{\frac{35}{4} + \frac{7}{9}} = \sqrt{\frac{343}{36}}$
$= \frac{\sqrt{49}\cdot\sqrt{7}}{\sqrt{36}}$
$= \frac{7\sqrt{7}}{6}$

28. $\frac{5\sqrt{5}}{6}$

80. $3\sqrt{2} - 3\sqrt{3}$

81. $\frac{\sqrt{5}}{3} + \frac{3}{\sqrt{5}} = \frac{\sqrt{5}}{3} + \frac{3\sqrt{5}}{5}$

$= \frac{5\sqrt{5}}{15} + \frac{9\sqrt{5}}{15}$

$= \frac{14\sqrt{5}}{15}$

82. $\sqrt{34} + \frac{1}{2}(4) - 2\sqrt{5} = \sqrt{34} + 2 - 2\sqrt{5}$

83. $\frac{1}{3}(3\sqrt{7}) + 3\sqrt{7} = \sqrt{7} + 3\sqrt{7}$

$= 4\sqrt{7}$

84. $\frac{1}{4}\sqrt{144}\cdot\sqrt{2} - \frac{1}{6}\sqrt{36}\cdot\sqrt{2} + \frac{1}{\sqrt{4}\sqrt{6}} = \frac{12}{4}\sqrt{2} - \frac{6}{6}\sqrt{2} + \frac{1}{2}\cdot\frac{1}{\sqrt{2}}$

$= 3\sqrt{2} - \sqrt{2} + \frac{\sqrt{6}}{12}$

$= 2\sqrt{2} + \frac{\sqrt{6}}{12}$

Section 15-5

78. A first approximation to $\sqrt{42}$ is 6.

Divide: $\frac{42}{6} = 7.00$

Average: $\frac{6 + 7.00}{2} = 6.50$ (The second approximation to $\sqrt{42}$)

Divide: $\frac{42}{6.5} \approx 6.462$

Average: $\frac{6.5 + 6.462}{2} = 6.481$ (The third approximation to $\sqrt{42}$.)

$(6.481)^2 = 42.003361$

$42 \approx 6.481$

79. A first approximation to $\sqrt{74}$ is 9.

Divide $\frac{74}{9} \approx 8.22$

Average: $\frac{9 + 8.22}{2} \approx 8.61$ The second approximation to $\sqrt{74}$)

Divide: $\frac{74}{8.6} \approx 8.605$

Average: $\frac{8.6 + 8.605}{2} \approx 8.602$

$(8.602)^2 = 73.994404$

$\sqrt{74} \approx 8.602$

[Note: $\frac{74}{8.6} \approx 8.6047$ so we rounded up to 8.605. However, since the average of 8.6 and 8.605, which 8.6025, is exactly half way between 8.602 and 8.603, we shall round down to 8.602. In this case, we do so, since we know that 8.6025 is already too large because we had rounded 8.6047 up to 8.605.]

80. A first approximation to $\sqrt{93}$ is 10.

Divide: $\frac{93}{10} = 9.30$

Average: $\frac{10 + 9.30}{2} = 9.65$ (The second approximation to $\sqrt{93}$.)

Divide: $\frac{93}{9.7} \approx 9.588$

Average: $\frac{9.7 + 9.588}{2} = 9.644$ (The third approximation to $\sqrt{93}$.)

$(9.644)^2 = 93.006736$

$\sqrt{93} \approx 9.644$

<u>Section 15-6</u>

78. $0.00470 = 47\cdot(10^{-4})$, so $\sqrt{0.00470} = \sqrt{47}\cdot\sqrt{10^{-4}} = \sqrt{47}(10^{-2})$

A good first approximation to $\sqrt{47}$ is 7.

Divide: $\frac{47}{7} \approx 6.71$

Average: $\frac{7 + 6.71}{2} \approx 6.86$ (The second approximation to $\sqrt{47}$ is 6.86.)

Divide: $\frac{47}{6.9} \approx 6.812$

Average: $\frac{6.9 + 6.812}{2} = 6.856$ (The third approximation to $\sqrt{47}$ is 6.856.)

Hence the third approximation to $\sqrt{.00470}$ is .06856.

79. $0.0470 = 4.7(10^{-2})$, so $\sqrt{0.047} = \sqrt{4.7}\cdot\sqrt{10^{-2}} = \sqrt{4.7}(10^{-1})$

A first approximation to $\sqrt{4.7}$ is 2.

Divide: $\frac{4.7}{2} = 2.35$

Average: $\frac{2 + 2.35}{2} = 2.18$ (The second approximation to $\sqrt{4.7}$.)

Divide: $\frac{4.7}{2.2} \approx 2.136$

Average: $\frac{2.2 + 2.136}{2} = 2.168$ (The third approximation to $\sqrt{4.7}$.)

Hence the third approximation to $\sqrt{0.047}$ is 0.2168.

80. $70260 = 7.026(10^4)$, so $\sqrt{70260} = \sqrt{7.026}\cdot\sqrt{10^4} = \sqrt{7.026}(10^2)$

A first approximation to $\sqrt{7.026}$ is 3.

Divide: $\frac{7.026}{3} = 2.342$

Average: $\frac{3 + 2.342}{2} = 2.671$

The second approximation to $\sqrt{7.026}$ is 2.67.

Divide: $\frac{7.026}{2.7} \approx 2.6022$

Average: $\frac{2.7 + 2.602}{2} = 2.651$ (The third approximation to $\sqrt{7.026}$.)

Hence the third approximation to $\sqrt{70260}$ is 265.1.

81. $1681 = 16.81(10^2)$, so $\sqrt{16.81} = \sqrt{16.81}\cdot\sqrt{10^2} = \sqrt{16.81}(10)$

A first approximation to $\sqrt{16.81}$ is 4.

Divide: $\frac{16.81}{4} \approx 4.20$

Average: $\frac{4 + 4.20}{2} = 4.10$ (The second approximation to $\sqrt{16.81}$)

Divide: $\frac{16.81}{4.1} = 4.1$ (Isn't that interesting?)

Since $(4.1)^2 = 16.81$, $\sqrt{16.81} = 4.1$

Hence $\sqrt{1681} = 41$. (Note that we do not use $\approx$ here, since this is not an approximation, but is exact.)

82. $0.1369 = 13.69(10^{-2})$, so $\sqrt{0.1369} = \sqrt{13.69}\cdot\sqrt{10^{-2}}$

A first approximation to $\sqrt{13.69}$ is 4.

Divide: $\frac{13.69}{4} \approx 3.42$

Average: $\frac{4 + 3.42}{2} = 3.71$ (The second approximation to $\sqrt{13.69}$.)

Divide: $\frac{13.69}{3.7} = 3.7$

Since $(3.7)^2 = 13.69$, $\sqrt{13.69} = 3.7$

Hence $\sqrt{0.1369} = 0.37$.

Section 15-7 Review

1. (a) $\sqrt{4\cdot 3} = 2\sqrt{3}$

   (b) $\frac{1}{6}$

   (c) $\sqrt{4\cdot 2a} = 2\sqrt{2a}$ and $a \geq 0$

   (d) $\frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}} = \frac{4\sqrt{3}}{3}$

   (e) $\sqrt{9\cdot 2x^2} = \sqrt{9}\sqrt{x^2}\sqrt{2} = 3|x|\sqrt{2}$

   (f) $\sqrt{6}\sqrt{6\cdot 4} = 6\cdot 2 = 12$

   (g) $\sqrt{4\cdot 2} - \sqrt{9\cdot 2} = 2\sqrt{2} - 3\sqrt{2} = -\sqrt{2}$

   (h) $a^3b^3c^2$

   (i) $\sqrt{2}\sqrt{2} + \sqrt{2}\sqrt{6} = 2 + \sqrt{12} = 2 + 2\sqrt{3}$

2. (a) $4\sqrt{3} - 5\sqrt{3} + 2\sqrt{3} = \sqrt{3}$

   (b) $\sqrt{100} = 10$

   (c) $2|(a + b)|$

   (d) $\sqrt{2}\cdot\sqrt{2} - \sqrt{2}\cdot\sqrt{2\cdot 3} = 2 - 2\sqrt{3}$

   (e) $\frac{1}{\sqrt{2}} - \frac{3}{2\sqrt{2}} = \frac{2 - 3}{2\sqrt{2}} = \frac{-1}{2\sqrt{2}}$
   (If a simple form with rational denominator is desired, $\frac{-1}{2\sqrt{2}} = \frac{-\sqrt{2}}{4}$.)

   (f) $\frac{\sqrt{3} + \sqrt{2}}{\sqrt{2}}$
   (With rational denominator, $\frac{\sqrt{6} + 2}{2}$.)

   (g) $\sqrt[3]{27}\cdot\sqrt[3]{2} = 3\sqrt[3]{2}$

3. (a) $\frac{\sqrt{3}}{6}$

   (b) $\frac{\sqrt{10}}{6}$

   (c) $\frac{\sqrt{7}}{2}$

   (d) $\sqrt{10}$

   (e) $\frac{\sqrt{3}}{4}$

   (f) $\frac{|x|\sqrt{2}}{3}$

   (g) $\frac{\sqrt{5x}}{x^2}$ and $x > 0$

   *(h) $\sqrt[3]{\frac{3}{4}} = \sqrt[3]{\frac{6}{8}} = \frac{\sqrt[3]{6}}{2}$

   *(i) $\sqrt[3]{\frac{1}{9a^2}} = \sqrt[3]{\frac{3a}{27a^3}} = \frac{\sqrt[3]{3a}}{3a}$ and $a \neq 0$

4. (a) $4|a|\sqrt{3} - |a|\sqrt{3} - |a|\sqrt{3} = 2|a|\sqrt{3}$

   (b) $\frac{\sqrt{3} + \sqrt{2}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{3}(\sqrt{2}) + \sqrt{2}\cdot\sqrt{2}}{2} = \frac{\sqrt{6} + 2}{2}$

   (c) $(\sqrt{3} - \sqrt{2})\sqrt{3} + (\sqrt{3} - \sqrt{2})\sqrt{2} = 3 - \sqrt{6} + \sqrt{6} - 2 = 1$

   (d) $\frac{2|m|\sqrt{q}}{q} + 7|m|q\sqrt{2q}$ where $q > 0$

   (e) $\sqrt{\frac{2}{3}\cdot\frac{5}{6}\cdot\frac{3}{2}} = \frac{\sqrt{5}}{\sqrt{6}} = \frac{1}{6}\sqrt{30}$

   (f) $\sqrt{18p^4} = 3p^2\sqrt{2}$ and $p \geq 0$

(g) $2\sqrt{a^2 + b^2}$

(h) $\dfrac{1}{\sqrt[3]{125 \cdot 2x}} = \dfrac{1}{5\sqrt[3]{2x}} \cdot \dfrac{\sqrt[3]{2^2x^2}}{\sqrt[3]{2^2x^2}} = \dfrac{\sqrt[3]{4x^2}}{10x}$ where $x \neq 0$

*(i) $\dfrac{1}{\sqrt[3]{2^2a}} \cdot \dfrac{\sqrt[3]{2a^2}}{\sqrt[3]{2a^2}} - \sqrt[3]{8 \cdot a^3 \cdot 2a^2} = \dfrac{\sqrt[3]{2a^2}}{2a} - 2a\sqrt[3]{2a^2} = \dfrac{\sqrt[3]{2a^2}}{2a} - \dfrac{4a^2\sqrt[3]{2a^2}}{2a}$

$= \dfrac{\sqrt[3]{2a^2}}{2a}(1 - 4a^2)$ where $a \neq 0$

5. (a) $\{4\}$
   *(b) $\{64\}$
   (c) $\{\sqrt{2}, -\sqrt{2}\}$
   (d) all m such that $-4 \leq m \leq 4$
   (e) $\{8\}$
   (f) $2|x| + |x| = 3$
   $3|x| = 3$
   $|x| = 1$
   The truth set is $\{-1, 1\}$.

6. (a) $\frac{1}{x} + \frac{2}{3x} = \frac{1}{3}$ for $x = 5$

   (b) $x + \sqrt{2} > \sqrt{2}$ for $x > 0$

7. $3900 = 39.00 \times 10^2$

   $\sqrt{3900} = \sqrt{39} \times \sqrt{10^2}$

   $\sqrt{39} \approx 6$; Divide: $\frac{39}{6} = 6.5$; Average: $\frac{6 + 6.5}{2} = 6.25 \approx 6.3$

   $\sqrt{39} \approx 6.3$; Divide: $\frac{39}{6.3} = 6.190$; Average: $\frac{6.3 + 6.190}{2} = 6.245$

   $\sqrt{39} \approx 6.245$

   $\sqrt{3900} \approx 6.245 \times 10 \approx 62.45$

8. 3900.0025

9. (a) 3^5
   (b) 6^2
   (c) $3^2 \cdot 2^3$
   (d) $3^3(1 + 1 + 1) = 3^3 \cdot 3 = 3^4$
   (e) $3^4(1 + 1 + 1) = 3^4 \cdot 3 = 3^5$
   (f) $3^2 + 2^2$

10. (a) $10^0 = 1$
    (b) $\frac{10^2}{10^1} = 10^1$
    (c) 10^{n+2}
    (d) $\frac{10^2}{10^{-2}} = 10^{2+2} = 10^4$
    (e) $10^{-3-5+3} = 10^{-5}$
    (f) 10^{6n}

Section 15-7 Review (continued)

11. (a) $\frac{4}{5}$

(b) $\frac{b^3}{2} \cdot \frac{6}{b} = 3b^2$

(c) $\frac{mn}{q^2} \cdot \frac{m^2}{2nq} = \frac{m^3}{2q^3}$

(d) $\frac{(\frac{1}{4} + \frac{1}{3})12}{(\frac{5}{12})12} = \frac{3 + 4}{5} = \frac{7}{5}$

(e) $\frac{(1 + \frac{1}{x})x}{(1 - \frac{1}{x})x} = \frac{x + 1}{x - 1}$

(f) $\frac{x + 3}{2} \cdot \frac{3}{x(x + 3)} = \frac{3}{2x}$

12. (a) $6(\frac{1}{3}x - 1) > 6(\frac{1}{2}x)$

$2x - 6 > 3x$

$-6 > x$

The set of real numbers less than -6.

(b) $3(\frac{8 - x}{3}) = 3(\frac{5x}{3})$

$8 - x = 5x$

$8 = 6x$

$\frac{8}{6} = x$

$\{\frac{4}{3}\}$

(c) $8y < 15$

$y < \frac{15}{8}$

The set of real numbers less than $\frac{15}{8}$.

(d) $20(|m| - \frac{3}{20}) = 20(\frac{1}{5}|m|)$

$20|m| - 3 = 4|m|$

$16|m| = 3$

$|m| = \frac{3}{16}$

$\{\frac{3}{16}, -\frac{3}{16}\}$

13. $n^2 - n + 41$ fails to give a prime for $n = 41$, since the sum of the last two terms is zero. This leaves n^2, which has n as a factor. If an algebraic sentence is true for the first 400 values of the variable, it is not certain that it is true for the 401st.

Section 15-7 Review (continued)

14. 200 is a good guess for the average since the weights cluster around 200.

$195 - 200 = -5$
$205 - 200 = 5$
$212 - 200 = 12$
$201 - 200 = 1$
$198 - 200 = -2$
$232 - 200 = 32$
$189 - 200 = -11$
$178 - 200 = -22$
$196 - 200 = -4$
$204 - 200 = 4$
$182 - 200 = -18$

Sum of the differences is -10.
Average of the differences is $-\frac{8}{11}$.

Adding this to 200 gives $199\frac{3}{11}$ for the team average.

* Let the n numbers to be averaged be represented by $a_1, a_2, a_3, \ldots, a_n$ where the "subscript" (the small number written to the lower right of a) in a_1 indicates that a_1 is the first number, the subscript in a_2 shows that a_2 is the second number, etc.

The average of n numbers $a_1, a_2, a_3, \ldots, a_n$ is:

$$\frac{a_1 + a_2 + a_3 + a_4 + \ldots + a_n}{n}$$

If g is the "guessed average," then the average of the differences is:

$$\frac{(a_1 - g) + (a_2 - g) + (a_3 - g) + \ldots (a_n - g)}{n}$$

$$= \frac{a_1 + a_2 + a_3 + \ldots + a_n - ng}{n}$$

(Since each term contains $-g$ and there are n terms, the sum of the g's is $-ng$.)

$$= \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} - \frac{ng}{n}$$

$$= \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} - g$$

When we add this average of the differences to our "guessed average" g, we have

$$\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} - g + g$$

$$= \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n},$$

and this is the average. Hence, the method works.

15. If the rat weighs x grams at the beginning of the experiment, it will weigh $\frac{5}{4}x$ grams after the rich diet and $\frac{3}{4}(\frac{5}{4}x)$ at the end of the experiment. Thus, the difference is $\frac{15}{16}x - x = -\frac{1}{16}x$ grams.

Section 16-6

1. $(z^2 + 8)^2$ [Perfect square.]

2. $(x^2 - 1)^2 - 9 = (x + 2)(x - 4)$ [Complete the square.]

3. $3(4x^2 - 9y^2) = 3(2x + 3y)(2x - 3y)$ [First find common factor (3). Then use difference of two squares.]

4. $(2x + 1)^2$ [Perfect square.]

5. Not possible. [Completing the square gives $(x + 1)^2 + 4$.]

6. $4(5 - x)$ [Common factor (4).]

7. $\big(3(a - 1) - 1\big)\big(3(a - 1) + 1\big) = (3a - 4)(3a - 2)$ [Difference of two squares.]

8. $(x^2 - 1)^2 = (x - 1)^2(x + 1)^2$ [$x^4 - 2x^2 + 1$ is a perfect square. $x^2 - 1$ is the difference of two squares.]

9. $5(c + d) + a(c + d) = (5 + a)(c + d)$ [Grouping.]

16. $\{0, -2\}$ [$x(x + 2) = 0$]

17. $\{5, -5\}$

18. $\{3, -5\}$

19. $\emptyset$

20. $\{-1, 5\}$ [First write: $x^2 - 4x - 5 = 0$. Then factor: $x^2 - 4x - 5 = (x - 2)^2 - 9 = (x + 1)(x - 5)$.]

Section 17-3

50.
$$\begin{aligned} a^2 + 3a + 1 &= a^2 + 3a + \tfrac{9}{4} - \tfrac{9}{4} + 1 \\ &= (a + \tfrac{3}{2})^2 - \tfrac{5}{4} \\ &= (a + \tfrac{3}{2} + \tfrac{\sqrt{5}}{2})(a + \tfrac{3}{2} - \tfrac{\sqrt{5}}{2}) \end{aligned}$$

51.
$$\begin{aligned} y^2 + y - 3 &= y^2 + y + \tfrac{1}{4} - \tfrac{1}{4} - 3 \\ &= (y + \tfrac{1}{2})^2 - \tfrac{13}{4} \\ &= (y + \tfrac{1}{2} + \tfrac{\sqrt{13}}{2})(y + \tfrac{1}{2} - \tfrac{\sqrt{13}}{2}) \end{aligned}$$

52. $x^2 - 5x - 2 = y^2 - 5x + \frac{25}{4} - \frac{25}{4} - 2$

$= (y - \frac{5}{2})^2 - \frac{33}{4}$

$= (y - \frac{5}{2} + \frac{\sqrt{33}}{2})(y - \frac{5}{2} - \frac{\sqrt{33}}{2})$

53. $y^2 + \frac{2}{3}y - 1 = y^2 + \frac{2}{3}y + \frac{1}{9} - \frac{1}{9} - 1$

$= (y + \frac{1}{3})^2 - \frac{10}{9}$

$= (y + \frac{1 + \sqrt{10}}{3})(y + \frac{1 - \sqrt{10}}{3})$

Section 17-5 Review

7. $x^2 + 6x + 9$

8. $x^2 - 4x + 4$

9. $x^2 + 2\sqrt{2}x + 2$

10. $a^2 + 2ab + b^2$

11. $x^2 - 2xy + y^2$

12. $x^2 - 2x + 1 - a^2$

13. $10,000 + 2(100) + 1 = 10201$

14. $4x^2 - 12xy + 9y^2$

15. $9a^2 + 24ab + 16b^2$

16. $(x + 2)(x - 2)$

17. not factorable over the integers

18. $(2t + 3)^2$

19. $(2z - 5)^2$

20. not factorable over the integers

21. not factorable over the integers

22. not factorable over the integers

23. $x^2 - 4x + bx - 4b$
$= x(x - 4) + b(x - 4)$
$= (x + b)(x - 4)$

24. $(9am + 6ab) + (12m + 8b)$
$= 3a(3m + 2b) + 4(3m + 2b)$
$= (3a + 4)(3m + 2b)$

25. $2a(a - 5b)^2$

26. $(z^2 + 8)^2$

27. $(n - 6)(n - 4)$

28. Not factorable. $z^2 - 2z + 18 = (z - 1)^2 + 17$

29. $-(x - 3)(x - 4)$

30. $-(x + 6)(x - 2)$

Section 17-5 Review (continued)

31. $-(x - 4)(x + 3)$

32. $(a - 8)^2$

33. Not factorable. $a^2 + 8a + 64 = (a + 4)^2 + 48$

34. $(a - 16)(a - 4)$

35. $(a - 8 + 8\sqrt{2})(a - 8 - 8\sqrt{2})$, or $\big(a - 8(1 - \sqrt{2})\big)\big(a - 8(1 + \sqrt{2})\big)$

36. $(d + \sqrt{2})(d - \sqrt{2})$

37. $(h - 13)(h + 13)$

38. Not factorable. $a^2 + 10a + 39 = (a + 5)^2 + 14$

39. $5a(a^2 - 3a + 6)$. This is the complete factorization, since
$$a^2 - 3a + 6 = (a - \tfrac{3}{2})^2 + \tfrac{15}{4}.$$

40. $7(x + 3)(x - 3)$

61. If n represents the number, then an open sentence is

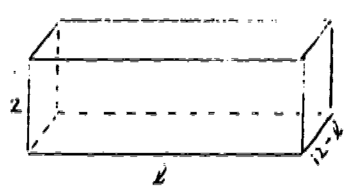
$$n^2 = 10n - 9$$

or $n^2 - 10n + 9 = 0$

The truth set is $\{1,9\}$. Thus the number is either 1 or 9.

Check: If the number is 1, its square is 1, and 1 is 9 less than 10(1).
If the number is 9, its square is 81, and 81 is 9 less than 10(9).

62. We can use the formula $V = \ell wh$ for the volume of a rectangular solid. If the perimeter of the base is 24 feet, then



the sum of the length in feet and the width in feet is 12 feet. If ℓ represents the length, then $12 - \ell$ represents the width, and an open sentence is

$$\ell(12 - \ell)2 = 70$$

The domain of ℓ is the set of positive real numbers less than 12.

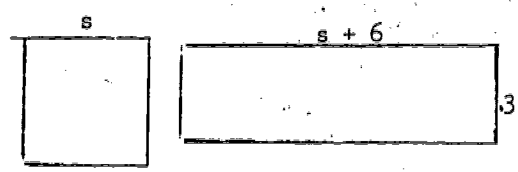
We have:
$$24\ell - 2\ell^2 = 70$$
$$-2\ell^2 + 24\ell - 70 = 0$$
$$-2(\ell^2 - 12\ell + 35) = 0$$
$$-2(\ell - 7)(\ell - 5) = 0$$

The truth set is $\{7,5\}$.

If $\ell = 7$, then $12 - \ell = 5$; that is, the length of the rectangle is 7 feet and the width is 5 feet. If $\ell = 5$, then $12 - \ell = 7$; that is, the length is 5 feet and the width is 7 feet.

In either case, the sides are 5 feet and 7 feet long, the perimeter of the base is 24 feet, and the volume is 70 cubic feet.

63. If s represents the length of a side of the square, then $s + 6$ represents the length of the rectangle. The area of the square is s^2.



The area of the rectangle is $3(s + 6)$.

An open sentence is

$$s^2 = 3(s + 6).$$

The domain of s is the set of positive real numbers.

We have:

$$s^2 = 3s + 18$$

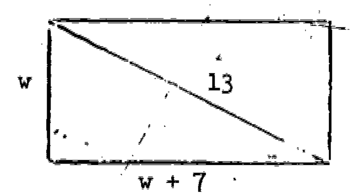
$$s^2 - 3 - 18 = 0$$

$$s - 6 = 0 \quad \text{or} \quad s + 3 = 0$$

Thus, the side of the square is 6 feet; the rectangle is 12 feet long and 3 feet wide.

Check: The area of the square is $6(6)$, or 36 square feet.
The area of the rectangle is $12(3)$, or 36 square feet.

64. If w is the number of inches in the width, then $w + 7$ is the number of inches in the length. Since the diagonal is 13 inches long, and the diagonal and two sides form a right triangle, an open sentence is:



$$w^2 + (w + 7)^2 = 13^2.$$

The domain of w is the set of positive real numbers. We have:

$$w^2 + w^2 + 14w + 49 = 169$$

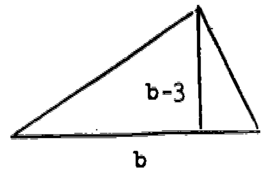
$$2w^2 + 14w - 120 = 0$$

$$w = -12 \quad \text{or} \quad w = 5$$

The rectangle is 5 inches wide.

Check: The length is $5 + 7$ or 12 inches, and $5^2 + 12^2 = 13^2$.

65. If b is the number of inches in the length of the base, then $b - 3$ is the number of inches in the altitude, and the area of the triangle is $\frac{1}{2}b(b - 3)$ square inches.



Since the area is 14 square inches, an open sentence is

$$\frac{1}{2}b(b - 3) = 14.$$

The domain of b is the set of positive real numbers.

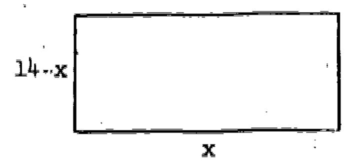
We have:

$$b(b - 3) = 28$$
$$b^2 - 3b - 28 = 0$$
$$b = 7 \quad \text{or} \quad b = -4$$

The length of the base is 7 inches.

Check: The altitude is $7 - 3$, or 4 inches.
The area is $\frac{1}{2}(7)(4)$, or 14 square inches.

66. If x is the number of inches in the length, then $14 - x$ is the number of inches in the width, and $x(14 - x)$ is the number of square inches in the area. Since the area is 24 square feet, an open sentence is



$$x(14 - x) = 24.$$

The domain of x is the set of positive real numbers.

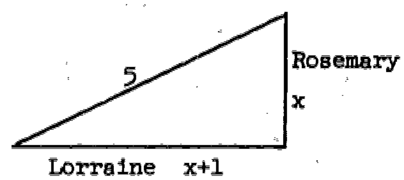
We have: $14x - x^2 - 24 = 0$

x is 12 or $x = 2$

The rectangle is 12 feet long and 2 feet wide, or it is 2 feet long and 12 feet wide.

Check: In either case, the area is 24 square feet.

67. If x is the number of miles Rosemary walked in 1 hour, then $x + 1$ is the number of miles Lorraine walked in 1 hour. Since we have a right triangle, an open sentence is:



$$x^2 + (x + 1)^2 = 5^2.$$

The domain of x is the set of positive real numbers.
We have:

$$2x^2 + 2x - 24 = 0$$
$$x = -4 \quad \text{or} \quad x = 3.$$

Thus Rosemary walked at the rate of 3 miles per hour, and Lorraine walked at the rate of 4 miles per hour.

Check: $3^2 + 4^2 = 9 + 16 = 25$
$5^2 = 25$

68. If n represents one number, then $15 - n$ represents the other number. Since the sum of the squares is 137, an open sentence is:

$$n^2 + (15 - n)^2 = 137$$
$$2n^2 - 30n + 88 = 0$$
$$2(n - 4)(n - 11) = 0$$

The truth set is $\{4,11\}$.
The numbers are 4 and $15 - 4$ or 11.
Check: $4^2 + 11^2 = 16 + 121 = 137.$

69. If n represents one number, then $n - 8$ represents the other number. Since their product is 84, an open sentence is:

$$n(n - 8) = 84$$
$$n^2 - 8n - 84 = 0$$

The truth set is $\{-6,14\}$.
The numbers are -6 and $-6 - 8 = -14$ or 14 and $14 - 8 = 6$.
Check: $-6(-14) = 84$, and $14(6) = 84$

70. If x is an odd number, then $x + 2$ is the next consecituve odd number.

An open sentence is

$$x(x + 2) = 15 + 4x$$
$$x^2 - 2x - 15 = 0.$$

The truth set is $\{5, -3\}$.

The numbers are 5 and 7, or -3 and -1.

Check: If the numbers are 5 and 7, their product is 35; $4(5) + 15$ is also 35.

If the numbers are -3 and -1, their product is 3; $4(-3) + 15$ is also 3.

71. Let x represent the number.

$$14x + x^2 = 11$$
$$x^2 + 14x - 11 = 0$$
$$(x + 7)^2 - 60 = 0$$
$$(x + 7)^2 - (2\sqrt{15})^2 = 0 \quad (\text{since } \sqrt{60} = 2\sqrt{15})$$
$$(x + 7 + 2\sqrt{15})(x + 7 - 2\sqrt{15}) = 0$$
$$\{-7 - 2\sqrt{15}, -7 + 2\sqrt{15}\} \text{ is the truth set.}$$

The number $-7 - 2\sqrt{15}$ satisfies the condition and so does $-7 + 2\sqrt{15}$.

72. $x^2 + 2\sqrt{3}x + 3 = (x + \sqrt{3})^2$. (Note that 3 is the square of half the coefficient of x.) Hence we have:

$$x^2 + 2\sqrt{3}x - 10 = 0$$
$$x^2 + 2\sqrt{3}x + 3 - 13 = 0$$
$$(x + \sqrt{3})^2 - (\sqrt{13})^2 = 0$$
$$(x + \sqrt{3} + \sqrt{13})(x + \sqrt{3} - \sqrt{13}) = 0$$

The truth set is $\{-\sqrt{3} - \sqrt{13}, -\sqrt{3} + \sqrt{13}\}$.

73. If m is an odd integer, then there is an <u>integer</u> n such that $m = 2n + 1$.

$$m^2 = (2n + 1)^2$$
$$= 4n^2 + 4n + 1$$
$$= 4(n^2 + n) + 1$$

Since n is an integer, $4(n^2 + n)$ is a multiple of 2. $\left(4(n^2 + n) = 2 \cdot 2(n^2 + n).\right)$ Hence $4(n^2 + n) + 1$ is odd.

*74. Referring to the previous problem: If n is odd, then

$$m^2 = (2n + 1)^2$$
$$= 4n(n + 1) + 1$$

n and $n + 1$ are consecutive integers. Hence one of them is even and the other odd. Hence $n(n + 1)$ is even. That is, there is an integer k for which $n(n + 1) = 2k$.

$$m^2 = 4 \cdot 2k + 1$$
$$m^2 - 1 = 4 \cdot 2k,$$

and $4 \cdot 2k$ is a multiple of 8.

Chapter 18 - DIVIDING POLYNOMIALS; RATIONAL EXPRESSIONS

18-1. Division of Polynomials

68. $$\begin{array}{r} 3a^2 + 7a - 11 \\ \underline{3a^2 - 6a + 9} \\ 13a - 20 \end{array}$$

69. $$\begin{array}{r} 12x^3 - 11x^2 \qquad + 3 \\ \underline{12x^3 \qquad\quad + 6x + 9} \\ -11x^2 - 6x - 6 \end{array}$$

70. $$\begin{array}{r} 14y^2 + 8y - 16 \\ \underline{-12y^2 + 3y \qquad} \\ 2y^2 + 11y - 16 \end{array}$$

71. $$\begin{array}{r} -6x + 8 \\ \underline{-6x - 1} \\ 9 \end{array}$$

18-2. Division of Polynomials, Concluded

18. $$\begin{array}{r|l} x - 5 \;\big)\; x^3 + 3x^2 - 38x - 10 & \underline{x^2 + 8x + 2} \\ \underline{x^3 - 5x^2} \qquad\qquad\quad & \\ 8x^2 - 38x \qquad\; & \\ \underline{8x^2 - 40x} \qquad\; & \\ 2x - 10 & \\ \underline{2x - 10} & \\ 0 & \end{array}$$

19. $$\begin{array}{r|l} x - 2 \;\big)\; 5x^2 + 3x - 3 & \underline{5x + 13} \\ \underline{5x^2 - 10x} \qquad & \\ 13x - 3 & \\ \underline{13x - 26} & \\ 23 & \end{array}$$

20. $$\frac{5x^2 + 3x - 3}{x - 2} = 5x + 13 + \frac{23}{x - 2}$$

24. $$\begin{array}{r|l} x - 6 \;\big)\; 2x^3 + 2x^2 + 0x + 5 & \underline{2x^2 + 14x + 84} \\ \underline{2x^3 - 12x^2} \qquad\qquad\quad & \\ 14x^2 + 0x \qquad\; & \\ \underline{14x^2 - 84x} \qquad\; & \\ 84x + 5 & \\ \underline{84x - 504} & \\ 509 & \end{array}$$

$$\frac{2x^3 + 2x^2 + 5}{x - 6} = 2x^2 + 14x + 84 + \frac{509}{x - 6}$$

18-3. Products and Quotients Involving Polynomials

44. $$\frac{\frac{x^2 - 9}{6}}{\frac{x^2 - 3x}{3x + 3}} = \frac{(x + 3)(x - 3)}{6} \cdot \frac{3(x + 1)}{x(x - 3)}$$

$$= \frac{(x + 1)(x + 3)}{2x}\text{, which may be written as } \frac{x^2 + 4x + 3}{2x}$$

Note we must have $x \neq 1$, $x \neq 0$, $x \neq 3$.

45. $$\frac{\frac{x^2 + x - 2}{x^2 - 4x + 4}}{\frac{x + 2}{x - 2}} = \frac{(x + 2)(x - 1)}{(x - 2)(x - 2)} \cdot \frac{x - 2}{x + 2} = \frac{x - 1}{x - 2}$$

We must have $x \neq 2$ and $x \neq -2$.

46. $$\frac{\frac{x^2 + 2x + 1}{x^2 - 1}}{\frac{x + 1}{x - 1}} = \frac{(x + 1)(x + 1)}{(x + 1)(x - 1)} \cdot \frac{x - 1}{x + 1} = 1$$

We must have $x \neq -1$, $x \neq 1$.

18-4. Rational Expressions

42. $$\frac{\frac{x}{2} - \frac{3}{x - 1}}{2 - \frac{x}{x + 1}} = \frac{\frac{x(x - 1) - 3(2)}{2(x - 1)}}{2 - x(x + 1)}$$

$$= \frac{\frac{x^2 - x - 6}{2(x - 1)}}{2 - x - x^2}$$

$$= \frac{\frac{(x - 3)(x + 2)}{2(x - 1)}}{(2 + x)(1 - x)}$$

$$= \frac{(x - 3)(x + 2)}{2(x - 1)} \cdot \frac{1}{(2 + x)(1 - x)}$$

$$= - \frac{x - 3}{2(x - 1)(x - 1)}$$

$$= \frac{3 - x}{2(x - 1)^2}$$

18-5. Summary and Review

1. (a) $$\frac{\frac{3x^2y^6}{20a^2b^2}}{\frac{7(xy^2)^3}{30(ab^2)^2}} = \frac{3x^2y^6}{20a^2b^2} \cdot \frac{30a^2b^4}{7x^3y^6}$$

$$= \frac{9b^2}{14x}$$

(b) $$\frac{3}{35a^2} + \frac{13}{25ab} - \frac{5}{7b^2} = \frac{15b^2 + 91ab - 125a^2}{175a^2b^2}$$

(c) $$\frac{2}{a^2 - ab} + \frac{3}{b^2 - ab} + \frac{4}{ab} = \frac{2}{a(a - b)} - \frac{3}{b(a - b)} + \frac{4}{ab}$$

$$= \frac{2b - 3a + 4(a - b)}{ab(a - b)}$$

$$= \frac{a - 2b}{ab(a - b)}$$

(d) $$\frac{x}{x^2 - 9} + \frac{2x-5}{x^2 - 4x+3} - \frac{3x}{x^2 + 2x-3} = \frac{x}{(x+3)(x-3)} + \frac{2x-5}{(x-3)(x-1)} - \frac{3x}{(x+3)(x-1)}$$

$$= \frac{x(x-1) + (2x-5)(x+3) - 3x(x-3)}{(x+3)(x-3)(x-1)}$$

$$= \frac{x^2 - x + 2x^2 + x - 15 - 3x^2 + 9x}{(x+3)(x-3)(x-1)}$$

$$= \frac{9x - 15}{(x+3)(x-3)(x-1)}$$

2. (a) $$\frac{x^3 - 4x^2 + x + 6}{x - 3} = x^2 - x - 2$$

(b) $$\frac{3x^4 + 14x^3 - 4x^2 - 11x - 2}{3x + 2} = x^3 + 4x^2 - 4x - 1$$

(c) $$\frac{x^3 - 1}{x + 1} = x^2 - x + 1 - \frac{2}{x + 1}$$

(d) $$\frac{x^5 - 1}{x - 1} = x^4 + x^3 + x^2 + x + 1$$

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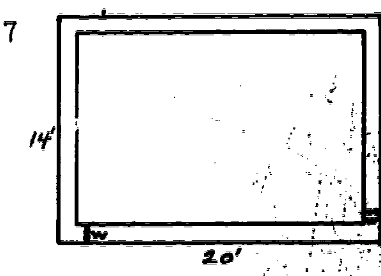
3. $$1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{2}}} = 1 + \cfrac{1}{1 + \frac{2}{3}}$$
$$= 1 + \frac{3}{5}$$
$$= \frac{8}{5}$$

4. If the width of the strip around the rug is w feet, then the number of feet in the length of the rug is $20 - 2w$, and the number of feet in the width of the rug is $14 - 2w$. Since the area of the rug is 24 square yards, or $24(9)$ square feet, an open sentence is:



$$(20 - 2w)(14 - 2w) = (24)(9), \quad 0 < w < 7$$
$$280 - 68w + 4w^2 = 216$$
$$4w^2 - 68w + 64 = 0$$
$$w^2 - 17w + 16 = 0$$
$$(w - 16)(w - 1) = 0$$

if $w - 1 = 0$, then $w = 1$

$w - 16 = 0$ has no solution such that $w < 7$.

Thus, the width of the strip is 1 foot.

5. (a) $x^2 - 22x - 48 = (x - 24)(x + 2)$

(b) $x^2 - y^2 - 4x - 4y = (x - y - 4)(x + y)$

(c) $3a^3b^5 - 6a^2b^3 + 12a^4b^4 = 3a^2b^3(ab^2 - 2 + 4a^2b)$

Chapter 19 - TRUTH SETS OF OPEN SENTENCES

19-1. Equivalent Equations

For Items 17-28 the truth sets are:

17. $\{6\}$
18. $\{6\}$
19. $\{6\}$
20. $\{\frac{7}{2}\}$
21. $\{-3\}$
22. $\emptyset$
23. $\{0,2\}$
24. $\{\frac{1}{15}\}$
25. $\{80\}$
26. $\{0\}$
27. $\{2,3\}$
28. Set of all real numbers.

38. Since $2(x^2 + 1)$ names a real non-zero number for all values of x, we may multiply both sides of $\frac{x^2}{x^2 + 1} = \frac{1}{2}$ by $2(x^2 + 1)$ and obtain the chain of equivalent equations:

$$2x^2 = x^2 + 1$$
$$x^2 = 1$$
$$x^2 - 1 = 0$$
$$(x + 1)(x - 1) = 0$$
$$x + 1 = 0 \text{ or } x - 1 = 0$$

The solution set is $\{-1,1\}$.

39. Since $x^2 + 5$ names a non-zero real number, we may multiply both sides of $\frac{x^2 + 5}{x^2 + 5} = 0$ by $x^2 + 5$ and obtain the equivalent equations:

$$x^2 + 5 = 0(x^2 + 5)$$
$$x^2 + 5 = 0$$

The solution set is $\emptyset$.

40. Since $x^2 + 5$ names a non-zero real number, we may multiply both sides of $\frac{x^2 + 5}{x^2 + 5} = 1$ by $x^2 + 5$ and obtain the equivalent equations:

$$x^2 + 5 = 1(x^2 + 5)$$
$$x^2 + 5 = x^2 + 5$$
$$0 = 0$$

The solution set of $0 = 0$ is the set of all real numbers. (Remember that although $0 = 0$ does not contain a variable, it is certainly a <u>true</u> sentence no matter what value the variable has. If you wish, you might consider $0 = 0$ as equivalent to $x + 0 = x + 0$ or $x = x$.)

74. $\{\sqrt{5}, -\sqrt{5}, \frac{1}{2}\}$

75. $\{\sqrt{7}, -\sqrt{7}, 2\sqrt{6}, -2\sqrt{6}\}$. We have written $\sqrt{24}$ as $2\sqrt{6}$.

76. $\{-5,0,5\}$

77. $x^3 + x = 2x^2$ is equivalent to $x^3 - 2x^2 + x = 0$

$$x(x^2 - 2x + 1) = 0$$
$$x(x - 1)^2 = 0 \quad \text{Truth set: } \{0,1\}$$

78. Completing the square, we see that $x^2 + 6x + 1 = 0$ is equivalent to

$$(x+3)^2 - 8 = 0$$

$$(x+3+2\sqrt{2})(x+3-2\sqrt{2}) = 0.$$

Solution set: $\{-3+2\sqrt{2}, -3-2\sqrt{2}\}$.

79. $\{0, 1, -\frac{1}{2}\}$

80. $\{-2, 2\}$

81. The given equation is equivalent to $x^2 = 3$ or $x - 1 = 0$.
The solution set is $\{\sqrt{3}, -\sqrt{3}, 1\}$.

82. The given equation is equivalent to $x^2 - 9 = 0$ or $x^4 + 2 = 0$. The solution set is $\{3, -3\}$.

83. We may rewrite the given equation in the form $x(3x - 1) = 2(3x - 1)$.
The truth set is $\{2, \frac{1}{3}\}$.

*96. Using the method indicated in Items 89-95, we see that a polynomial which has the value 0 for the given values of the variable is:

$$(x - \tfrac{2}{3})(x + \tfrac{1}{2})(x - \tfrac{5}{6}).$$

In order to write a polynomial with integer coefficients, we multiply the polynomial above by 36. (Do you see why we choose 36?)

$$36(x - \tfrac{2}{3})(x + \tfrac{1}{2})(x - \tfrac{5}{6}) = 3(x - \tfrac{2}{3})2(x + \tfrac{1}{2})6(x - \tfrac{5}{6})$$

$$= (3x - 2)(2x + 1)(6x - 5)$$

Multiplying and combining terms we obtain $3x^3 - 36x^2 - 7x + 10$ as a polynomial with the desired properties.

You should notice that the polynomial $36x^3 - 36x^2 - 7x + 10$ is 36 times the polynomial $(x - \frac{2}{3})(x + \frac{1}{2})(x - \frac{5}{6})$.

19-2. Fractional Equations

21. $x + \frac{1}{x} = 2$ is equivalent to

$$x^2 + 1 = 2x \quad \text{and} \quad x \neq 0$$

$$x^2 - 2x + 1 = 0 \quad \text{and} \quad x \neq 0$$

$$(x - 1)^2 = 0 \quad \text{and} \quad x \neq 0$$

The only solution is 1. The solution set is $\{1\}$.

29. R. $\frac{s - 2}{s} + \frac{3}{s^2} = 1$

$s^2(\frac{s - 2}{s} + \frac{3}{s^2}) = 1(s^2)$ and $s \neq 0$

$s^2 - 2s + 3 = s^2$ and $s \neq 0$

$-2s + 3 = 0$ and $s \neq 0$

$s = \frac{3}{2}$

The solution set is $\{\frac{3}{2}\}$.

S. $\frac{1 - y}{1 + y} + \frac{1 + y}{1 - y} = 0$

$(1 + y)(1 - y)(\frac{1 - y}{1 + y} + \frac{1 + y}{1 - y}) = 0(1 + y)(1 - y)$ and $y \neq 1$, $y \neq -1$

$(1 - y)^2 + (1 + y)^2 = 0$ and $y \neq 1$, $y \neq -1$

Since $y \neq 1$ and $y \neq -1$, both $(1 - y)^2$ and $(1 + y)^2$ are <u>positive</u>. The solution set is ϕ.

T. $\frac{1}{t} = \frac{1}{t - 1}$

$t(t - 1)(\frac{1}{t}) = (\frac{1}{t - 1})t(t - 1)$ and $t \neq 0$, $t \neq 1$

$t - 1 = t$ and $t \neq 0$, $t \neq 1$

$-1 = 0$

The solution set is ϕ.

U. $\frac{1 - y}{1 + y} - \frac{1 + y}{1 - y} = 0$

$(1 + y)(1 - y)(\frac{1 - y}{1 + y} - \frac{1 + y}{1 - y}) = 0(1 + y)(1 - y)$ and $y \neq 1$, $y \neq -1$

$(1 - y)^2 - (1 + y)^2 = 0$ and $y \neq 1$, $y \neq -1$

$(1 - 2y + y^2) - (1 + 2y + y^2) = 0$ and $y \neq 1$, $y \neq -1$

$-4y = 0$ and $y \neq 1$, $y \neq -1$

The solution set is $\{0\}$.

28. $(\frac{x-1}{x+1})^2 = 4$ is equivalent to $(x-1)^2 = 4(x+1)^2$ and $x \neq -1$

$$x^2 - 2x + 1 = 4x^2 + 8x + 4$$

$$3x^2 + 10x + 3 = 0$$

$$(3x+1)(x+3) = 0 \quad \text{and} \quad x \neq -1$$

Truth set: $\{-3, -\frac{1}{3}\}$.

32. We wish to find the solution of the equation $\frac{1}{3.2} + \frac{1}{r} = \frac{1}{2.4}$. Notice that the "sum of the reciprocals" is not equal to the "reciprocal of the sum".

Solving $\frac{1}{3.2} + \frac{1}{r} = \frac{1}{2.4}$

$$2.4r + (3.2)(2.4) = 3.2r$$

$$.8r = (3.2)(2.4)$$

$$r = 9.6$$

(In this problem the domain of r is the set of positive real numbers.)

19-3. Squaring Both Sides of an Equation

39. "Squaring" $\sqrt{x^2 - 16} = 8 - x$, we have

$$x^2 - 16 = 64 - 16x + x^2$$

$$16x = 80$$

$$x = 5$$

Check: Left member: $\sqrt{25 - 16} = \sqrt{9} = 3$

Right member: $8 - 5 = 3$

Truth set: $\{5\}$

40. "Squaring" $\sqrt{x^2 - 16} = x - 8$, we have

$$x^2 - 16 = x^2 - 16x + 64$$

$$16x = 80$$

$$x = 5$$

Check: Left member: $\sqrt{25 - 16} = \sqrt{9} = 3$

Right member: $5 - 8 = -3$

Truth set: $\emptyset$

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41. "Squaring" $\sqrt{x^2} = x$, we have

$$x^2 = x^2,$$ which is true for all real numbers.

However, $\sqrt{x^2}$ is non-negative, so x must be non-negative.

Truth set of $\sqrt{x^2} = x$: Set of all non-negative real numbers.

42. "Squaring" $\sqrt{x} = 2 - x$, we have

$$x = 4 - 4x + x^2$$
$$0 = 4 - 5x + x^2$$
$$0 = (4 - x)(1 - x)$$ which has solutions 1 and 4.

Check: If x is 1, the left side is $\sqrt{1} = 1$;
the right side is $2 - 1 = 1$.

If x is 4, the left side is $\sqrt{4} = 2$;
the right side is $2 - 4 = -2$.

Truth set of $\sqrt{x} = 2 - x$: $\{1\}$.

43. "Squaring" $\sqrt{2x} = 1 + x$, we have

$$2x = 1 + 2x + x^2$$
$$0 = 1 + x^2$$

Truth set: $\emptyset$

44. "Squaring" $3\sqrt{x + 13} = x + 9$, we have

$$9(x + 13) = x^2 + 18x + 81$$
$$9x + 117 = x^2 + 18x + 81$$
$$0 = x^2 + 9x - 36$$
$$0 = (x - 3)(x + 12)$$

Checking reveals that 3 is a solution of the original equation, but **that** -12 **is not.**

Truth set of $3\sqrt{x + 13} = x + 9$ is $\{3\}$.

46. $\{9\}$. **Note:** 1 is not a solution of the original equation.

47. $\{0\}$.

48. $\{0, -1\}$. **Be sure to check that both** 0 and -1 are solutions.

49. $\{\frac{3}{4}\}$. **Check!**

56.
$$\begin{aligned} |2x| &= x + 1 \\ 4x^2 &= x^2 + 2x + 1 \\ 3x^2 - 2x - 1 &= 0 \\ (3x + 1)(x - 1) &= 0 \end{aligned}$$

Truth set of $|2x| = x + 1$ is $\{-\frac{1}{3}, 1\}$.

57.
$$\begin{aligned} x - |x| &= 1 \\ x - 1 &= |x| \\ x^2 - 2x + 1 &= x^2 \\ -2x + 1 &= 0 \end{aligned}$$

Truth set of $x - |x| = 1$ is $\emptyset$.

58.
$$\begin{aligned} 2x &= |x| + 1 \\ 2x - 1 &= |x| \\ 4x^2 - 4x + 1 &= x^2 \\ 3x^2 - 4x + 1 &= 0 \\ (3x - 1)(x - 1) &= 0 \end{aligned}$$

Truth set of $2x = |x| + 1$ is $\{1\}$.

59.
$$\begin{aligned} |x - 3| &= 4 \\ (x - 3)^2 &= 16 \\ x^2 - 6x + 9 &= 16 \\ x^2 - 6x - 7 &= 0 \\ (x + 1)(x - 7) &= 0 \end{aligned}$$

Truth set of $|x - 3| = 4$ is $\{-1,7\}$.

60. An open sentence for this problem is $|x - 3| = x + 2$.
Solving, we have,
$$\begin{aligned} (x - 3)^2 &= (x + 2)^2 \\ x^2 - 6x + 9 &= x^2 + 4x + 4 \\ 10x &= 5 \\ x &= \frac{1}{2} \end{aligned}$$

Check in original equation!

19-4. Inequalities

1. $x + 3 < 9$. Subtracting 3 (adding -3) to both sides, we have $x < 3$.

2. $\frac{x}{7} > 4$. Multiplying by the positive number 7, we have $x > 28$.

3. $-3x > 54$. Multiplying by the negative number $-\frac{1}{3}$, we have $x < -18$. (Notice the change in the order.)

4. $3x - 2 > x + 4$ is equivalent to

$$3x - x > 4 + 2$$
$$2x > 6$$
$$x > 3.$$

11.

Sentences	Directions
$4 - \frac{x}{2} > x - \frac{1}{2}$	First, multiply by 2, order is not changed.
$8 - x > 2x - 1$	Now subtract 8 and $2x$.
$-x - 2x > -1 - 8$	
$-3x > -9$	Finally, divide by -3, changing order.
$x < 3.$	

The truth set is the set of all numbers less than 3.

12.

$x < 3$	Multiply by -3: change order.
$-3x > -9$	Write $-3x$ as $-x - 2x$; -9 as $-1 - 8$.
$-x - 2x > -1 - 8$	Add $2x$ and 8.
$8 - x > 2x - 1$	Divide by 2, order unchanged.
$4 - \frac{x}{2} > x - \frac{1}{2}$	

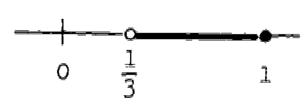
23. $1 < 4x + 1 < 2$

$1 < 4x + 1$	and	$4x + 1 < 2$
$0 < 4x$	and	$4x < 1$
$0 < x$	and	$x < \frac{1}{4}$

Graph: 

24.

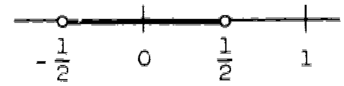
$4t - 4 \leq 0$	and	$1 - 3t < 0$
$4t \leq 4$	and	$-3t < -1$
$t \leq 1$	and	$t > \frac{1}{3}$

Graph: 

25. $-1 < 2t < 1$

$-1 < 2t$ and $2t < 1$

$-\frac{1}{2} < t$ and $t < \frac{1}{2}$

Graph: 

Notice that this is the graph of $|t| < \frac{1}{2}$.

26. $6y + 3 < 0$ or $6y - 3 > 0$

$6y < -3$ or $6y > 3$

$y < -\frac{1}{2}$ or $y > \frac{1}{2}$

Graph: 

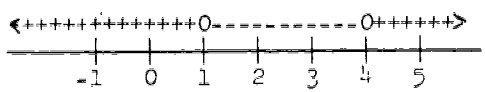
Notice that this is the graph of $|y| > \frac{1}{2}$.

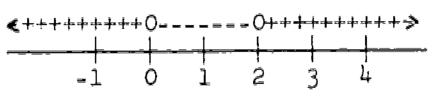
80. Since $-\dfrac{1}{x^2 + 2}$ is negative for all values of x,

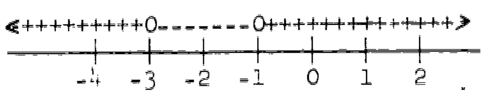
$-\dfrac{2}{x^2 + 2} \geq -1$ is equivalent to

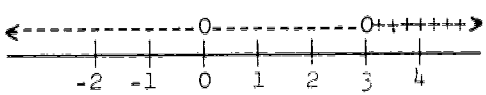
$2 \leq x^2 + 2$

$0 \leq x^2$, which is true for all real numbers x.

81. 

82. 

83. 

84. 

Notice that $x^2 > 0$ unless $x = 0$.

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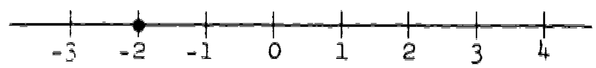
19-5. <u>Summary</u> <u>and</u> <u>Review</u>

1.	{3}	6.	{-2,2}	11.	{0,1,2}	16.	{2,-4}
2.	{5}	7.	$\{\frac{4}{25}\}$	12.	{-4}	17.	$\{\frac{1}{2}\}$
3.	{5}	8.	{-3}	13.	∅	18.	∅
4.	∅	9.	{3,6}	14.	∅	19.	Set of all real numbers except -1.
5.	{-2}	10.	{-1,1}	15.	{2}	20.	Set of all real numbers.

21. (a) The truth set is {-2}.

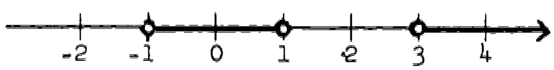
The graph: 

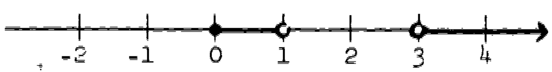
*(b) The truth set is the set of all values of x such that $x < -2$ or $x > 2$.

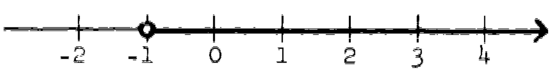
The graph: 

*(c) Same as *(b).

(d) Same as *(b).

*22. (a) 

(b) 

(c) 

23. $\frac{300}{30} = 10$ hours one way.

$\frac{300}{20} = 15$ hours returning.

Since the average rate for the whole trip must involve <u>total</u> distance (600 miles) and <u>total</u> time (25 hours), the average rate is $\frac{600}{25}$ or 24 miles per hour.

*24. If d is the distance in miles one way $(d > 0)$ and the rate is r miles per hour, the time one way is $\frac{d}{r}$ hours. On the return, if the rate is q miles per hour, the time is $\frac{d}{q}$ hours. The total distance, $2d$ miles, divided by the total time, $\frac{d}{r} + \frac{d}{q}$ hours, will be

$$\frac{2d}{\frac{d}{r} + \frac{d}{q}} = \frac{2d}{\frac{d}{r} + \frac{d}{q}} \cdot \frac{rq}{rq} = \frac{2drq}{dq + dr} = \frac{2drq}{d(q + r)} = \frac{2rq}{q + r} \cdot \frac{d}{d}$$

$$= \frac{2rq}{q + r} \text{ miles per hour, } q \neq 0, \quad r \neq 0.$$

Notice that the average rate does not depend on the distance traveled.

25. If n is the number of mph for the faster car, then $n - 4$ is the number of mph for the second. Then $\frac{360}{n}$ is the number of hours during which the faster travels, and $\frac{360}{n - 4}$ is the number of hours during which the slower travels. Hence,

$$\frac{360}{n} = \frac{360}{n - 4} - 1, \quad \text{if } n \neq 0, \quad n \neq 4, \quad n > 0.$$

$$\begin{aligned} 360(n - 4) &= 360n - n(n - 4) \\ 360n - 1440 &= 360n - n^2 + 4n \\ n^2 - 4n - 1440 &= 0 \\ (n + 36)(n - 40) &= 0 \end{aligned}$$

Since $n > 0$, 40 is the only solution. The rate of the faster car is 40 mph. Since $40 - 4 = 36$, the rate of the slower car is 36 mph.

26. If x is the number of units in the length of the shorter leg, then $2x + 2$ is the number of units in the longer leg. Hence, by the Pythagorean relationship,

$$x^2 + (2x + 2)^2 = 13^2, \quad 0 < x < 13.$$

$$\begin{aligned} x^2 + 4x^2 + 8x + 4 &= 169 \\ 5x^2 + 8x - 165 &= 0 \\ (5x + 33)(x - 5) &= 0 \end{aligned}$$

$$5x + 33 = 0 \quad \text{or} \quad x - 5 = 0$$

$5x + 33 = 0$ has no positive solution.

Thus, the truth set is $\{5\}$. The shorter leg is 5 units in length, and the longer leg is 12 units.

*27. $|x - 5|^2 \geq 9$

$|x - 5| \geq 3$, since $|x - 5| \geq 0$ for all values of x.

$x - 5 \geq 3$ or $x - 5 \leq -3$

$x \geq 8$ or $x \leq 2$.

The truth set is the set of all x such that $x \geq 8$ or $x \leq 2$.

*28. While the hour hand travels over a number of minute markings, x, the minute hand travels over $12x$ of these units. Since the hour hand is at 3 o'clock position, it has a 15-unit "head-start" over the minute hand at the time 3:00. Thus,

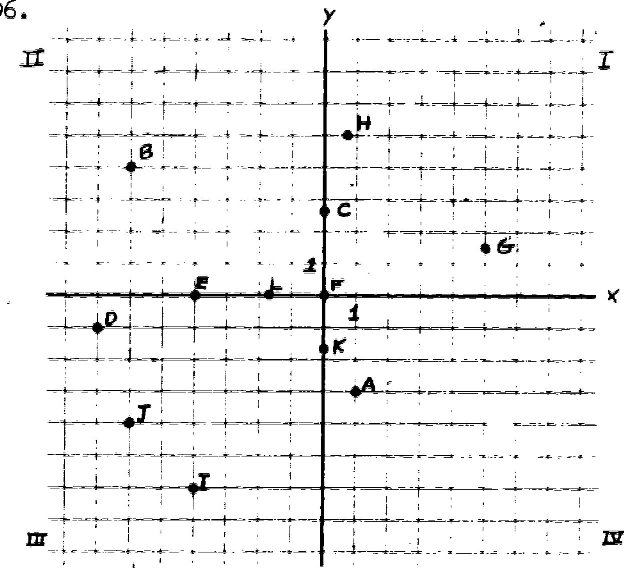
$$12x = x + 15,$$
$$11x = 15,$$
$$x = \frac{15}{11}.$$

Thus, the truth set of the equation is $\{\frac{15}{11}\}$. While the hour hand is traveling $\frac{15}{11}$ units, the minute hand travels $12(\frac{15}{11})$ or $16\frac{4}{11}$ units. Therefore, the hands will be together at $16\frac{4}{11}$ minutes after 3 o'clock.

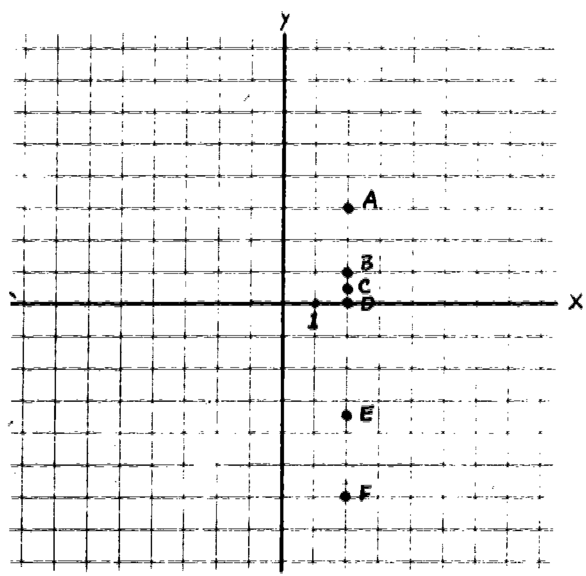
Chapter 20 - THE GRAPH OF $Ax + By + C = 0$

20-1. The Real Number Plane

96.



105. and
106.



109.

Possible points:

$(-9,5)$

$(-6\frac{1}{2}, 5)$

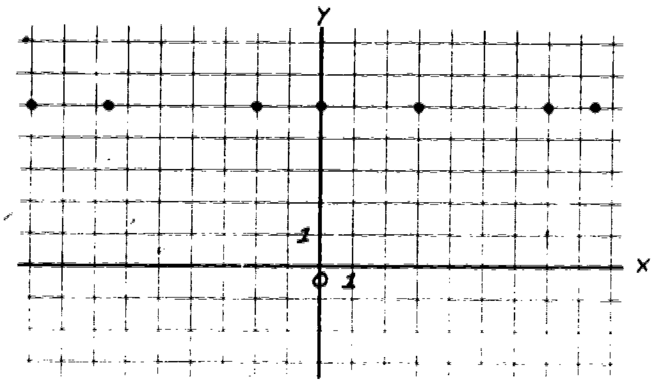
$(-2,5)$

$(0,5)$

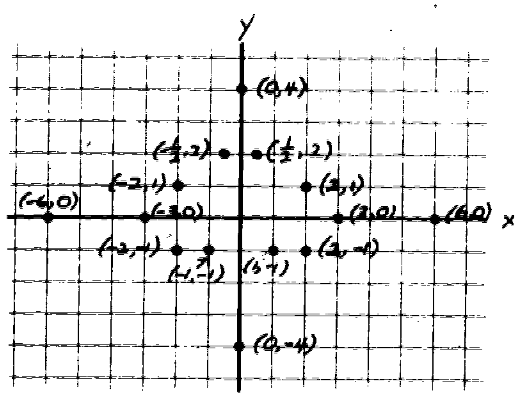
$(3,5)$

$(7,5)$

$(8\frac{1}{2}, 5)$



*110.



(a) $(2,1)$ goes to $(-2,1)$; $(3,0)$ goes to $(-3,0)$

$(2,-1)$ goes to $(-2,-1)$; $(-6,0)$ goes to $(6,0)$

$(-\frac{1}{2},2)$ goes to $(\frac{1}{2}, 2)$; $(0,4)$ goes to $(0,4)$

$(-1,-1)$ goes to $(1,-1)$; $(0,-4)$ goes to $(0,-4)$

*110. (continued)

(b) (-2,1) goes to (2,1); (-3,0) goes to (3,0)

(-2,-1) goes to (2,-1); (6,0) goes to (-6,0)

$(\frac{1}{2}, 2)$ goes to $(-\frac{1}{2}, 2)$; (0,4) goes to (0,4)

(1,-1) goes to (-1,-1); (0,-4) goes to (0,-4)

(c) (c,-d) goes to (-c,-d)

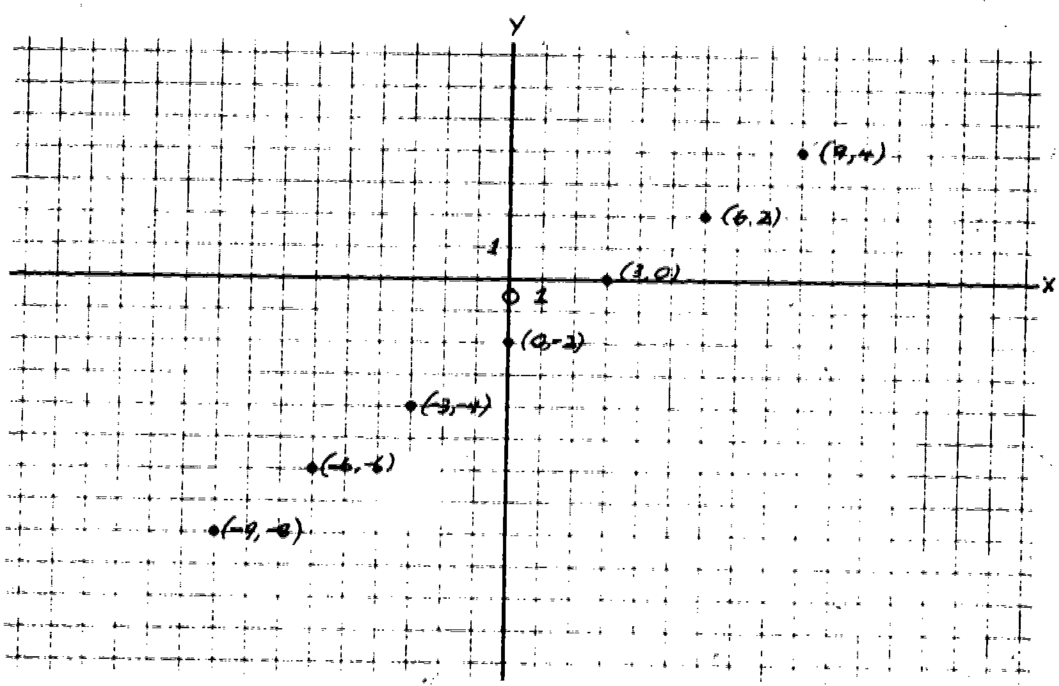
(d) (-c,d) goes to (c,d)

(e) (-c,d) goes to (c,d)

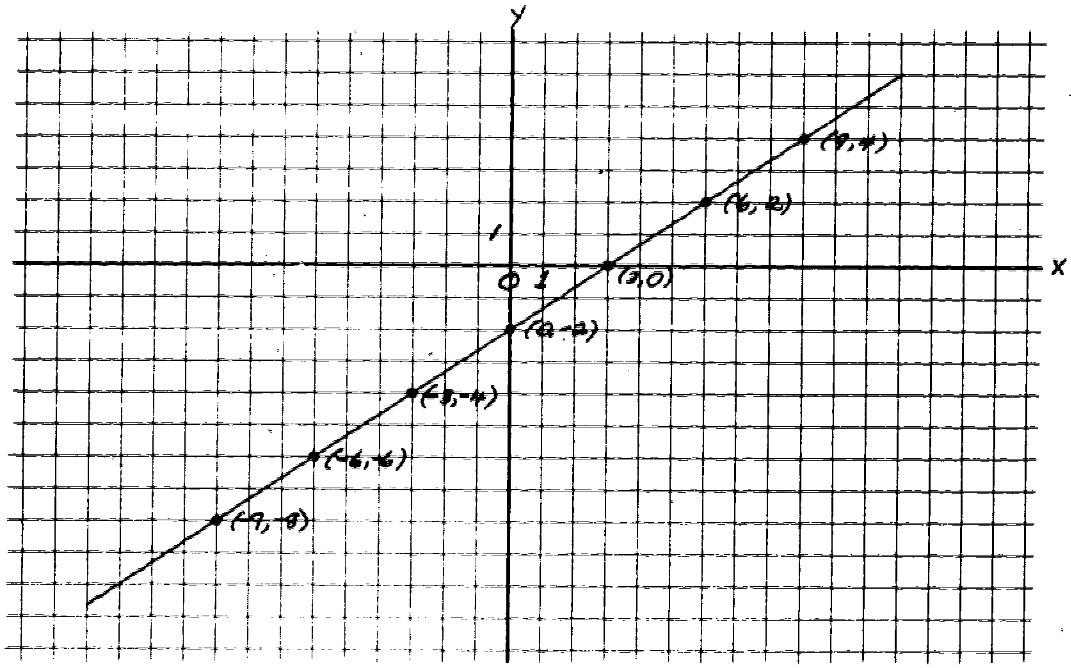
(f) Any point on the y-axis goes to itself since the first coordinate for a point on the y-axis is 0, and -0 = 0.

20-2. The y-Form of the Equation of a Line

Graph for Item 38



Graph for Item 39 and reference for Items 40-46.



66. $2y + 5x + 7 = 0$

$$2y = -5x - 7$$

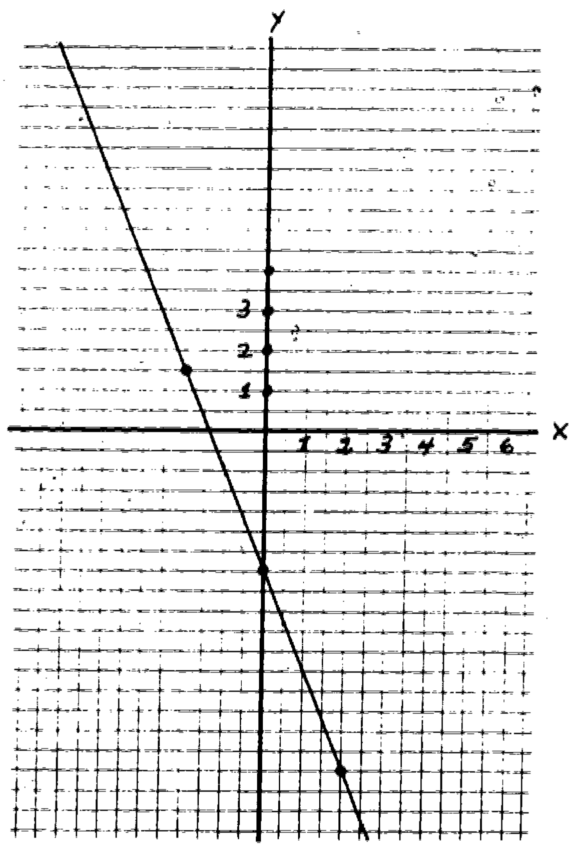
$$y = \frac{-5}{2}x - \frac{7}{2}$$

The y-form is $y = -\frac{5}{2}x - \frac{7}{2}$.

We may use multiples of 2 as values of the abscissa.

x	-6	-4	-2	0	2	4	6
y	$\frac{23}{2}$	$\frac{13}{2}$	$\frac{3}{2}$	$-\frac{7}{2}$	$-\frac{17}{2}$	$-\frac{27}{2}$	$-\frac{37}{2}$

Graph is:



74. Line (a) includes all possible points such that each has its abscissa equal to the opposite of the ordinate.

Line (b) includes those points such that each has ordinate twice the abscissa.

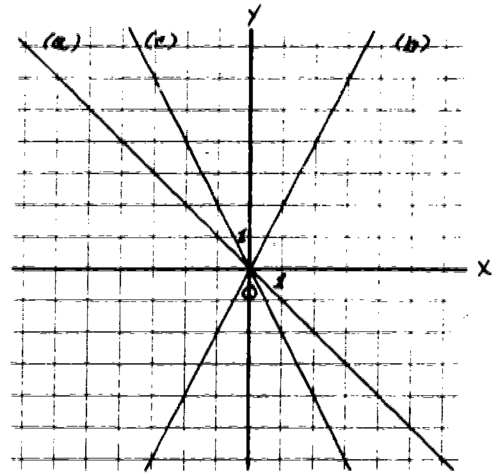
Line (c) includes the points such that each has ordinate that is the opposite of twice the abscissa.

All of these graphs are lines, and all pass through the origin. Their equations are:

(a) $y = -x$

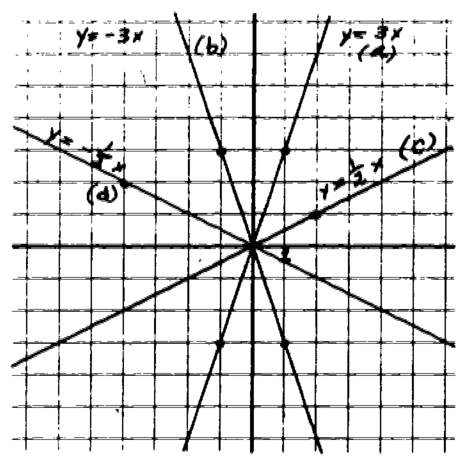
(b) $y = 2x$

(c) $y = -2x$

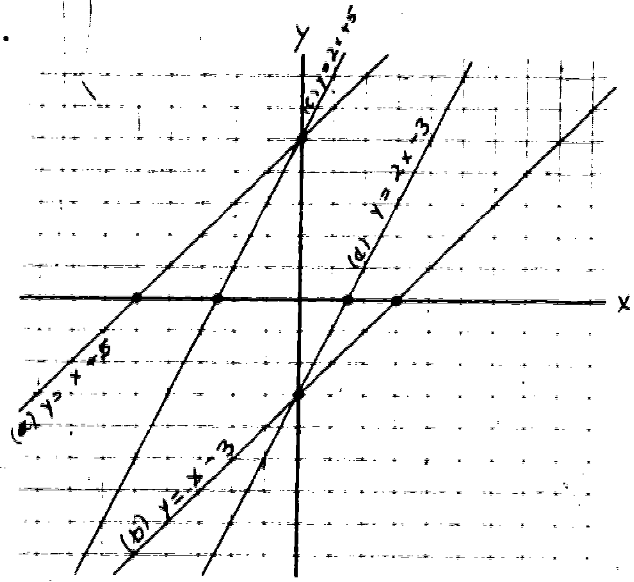


75. (Also, reference for Items 78-82.)

All of the graphs are lines through the origin. The graph of (a) rises as it goes from left to right, while the graph of (d) descends. The same pattern applies to the graphs of (c) and (b).

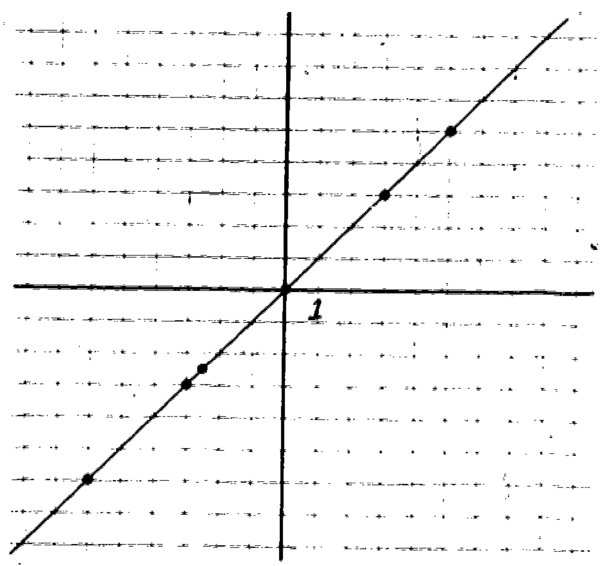


81.



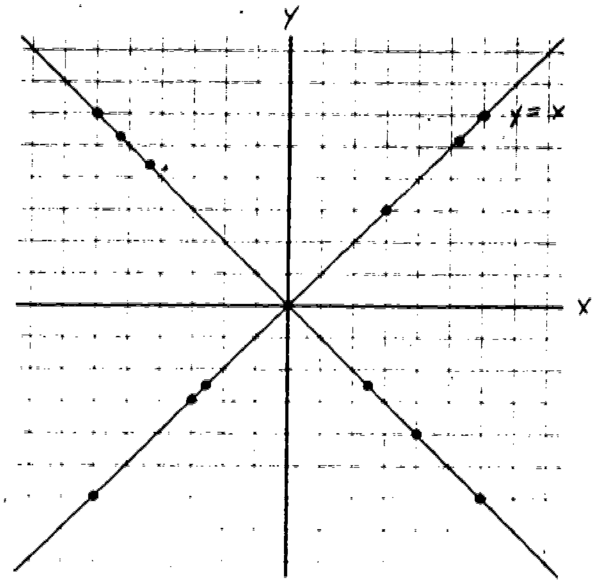
20-3. Definition of Slope and y-Intercept

2.

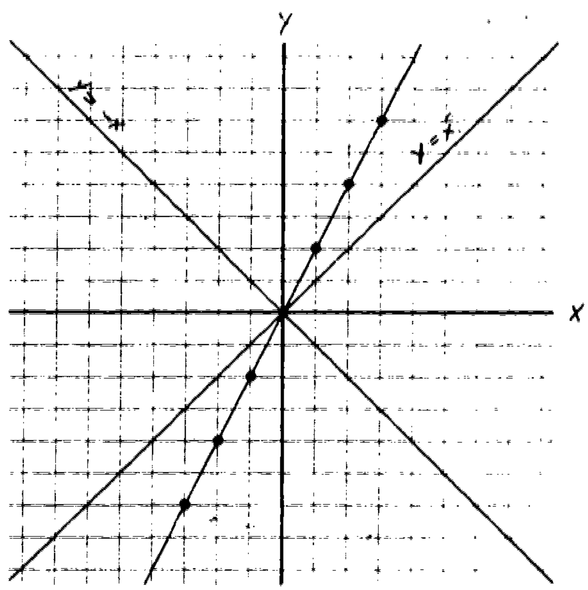


51?

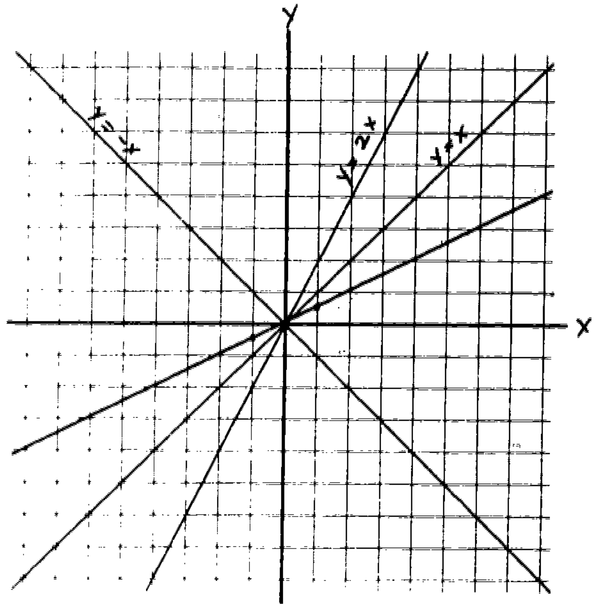
9.



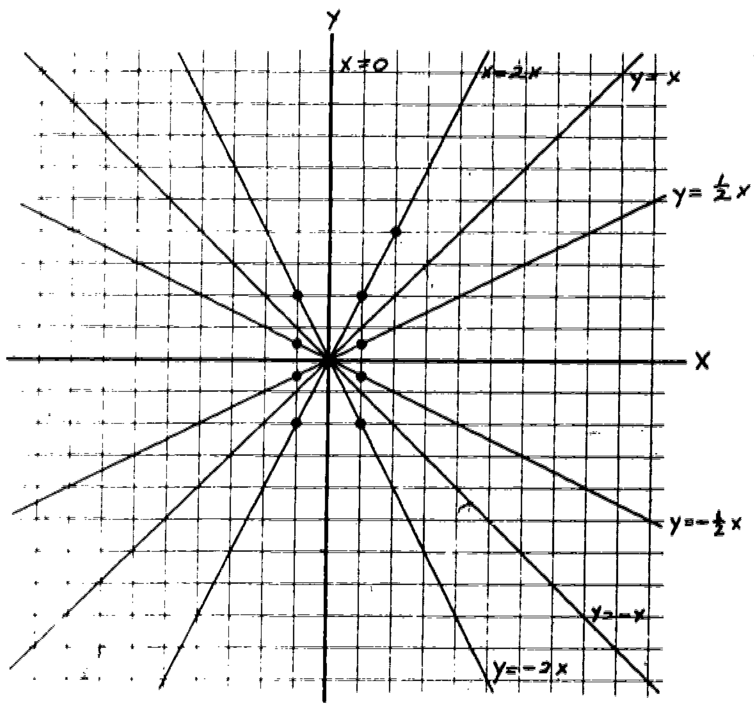
16.



20.

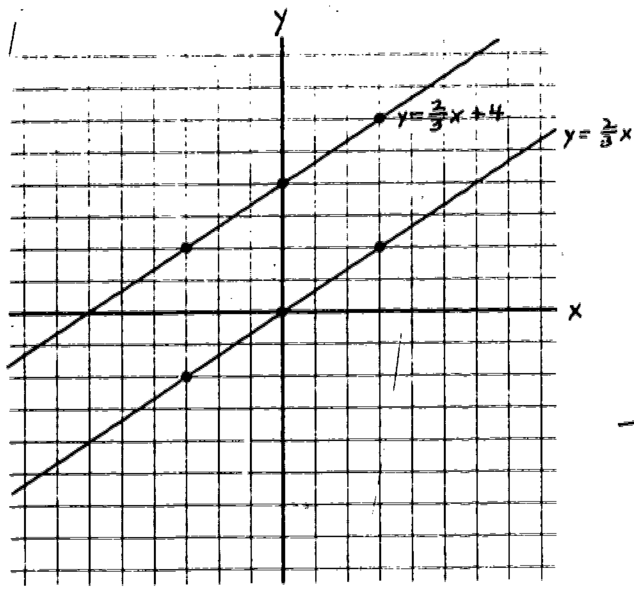


2_.

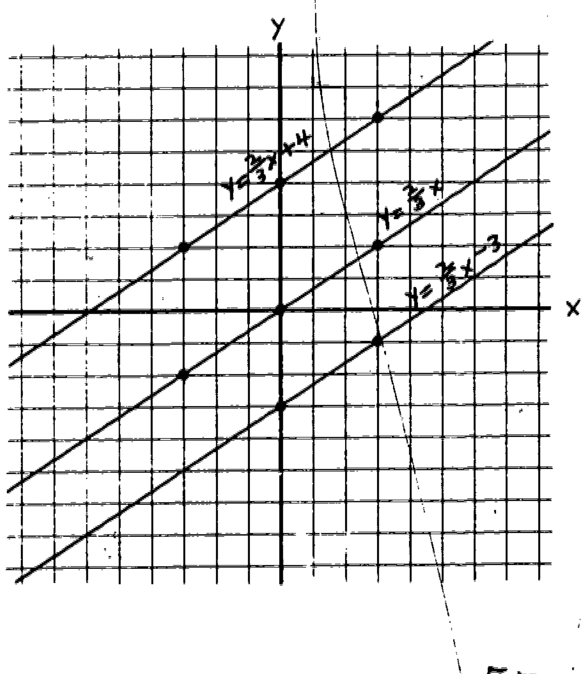


51_

50.



53.



78. Slope is 1, y-intercept is $(0,207)$.

79. Slope is -1, y-intercept is $(0, \frac{7}{2})$.

80. Slope is 1, y-intercept is $(0, \frac{11}{3})$.

81. Slope is -3, y-intercept is $(0,1)$.

82. Slope is $-\frac{1}{2}$, y-intercept is $(0, \frac{5}{4})$.

83. Slope is $\frac{7}{3}$, y-intercept is $(0, \frac{2}{3})$.

84. Slope is 0, y-intercept is $(0,9)$.

85. Slope undefined, no y-intercept (vertical line $x = -5$ does not intersect y-axis).

86. Slope is m, y-intercept is $(0,b)$.

20-4. Applications of the Slope and Intercept

36. Slope is $\frac{-3 - 2}{-7 - 6} = \frac{5}{13}$.

37. Slope is 0 since the ordinates are equal; i.e., this is a horizontal line: $\frac{3 - 3}{-7 - 8} = \frac{0}{-15} = 0$.

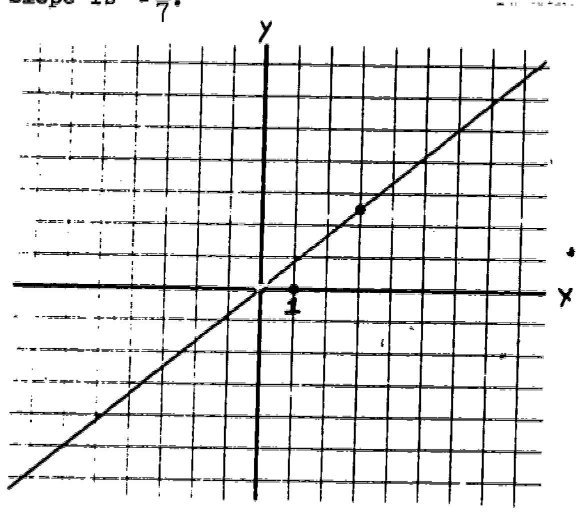
38. Slope is -2.

39. Slope undefined. This is a vertical line since the abscissas are equal.

40. Slope is $\frac{1}{3}$; $\frac{-2 - 0}{-6 - 0} = \frac{1}{3}$.

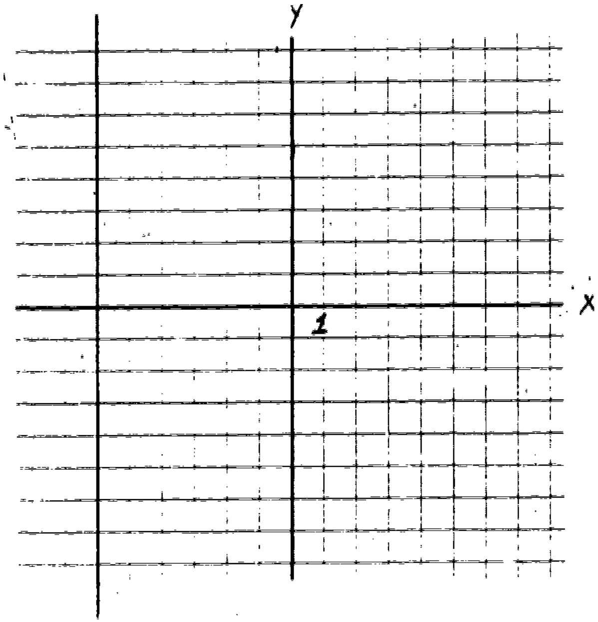
41. Slope is $-\frac{4}{7}$.

78.



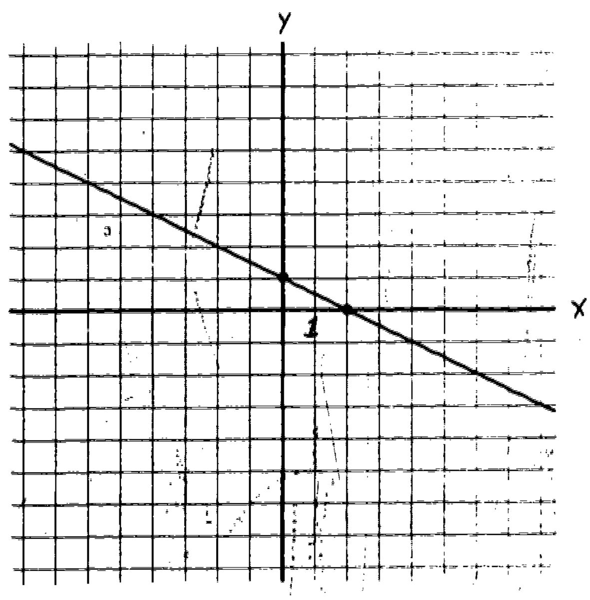
$y = \frac{5}{6}x$

79.



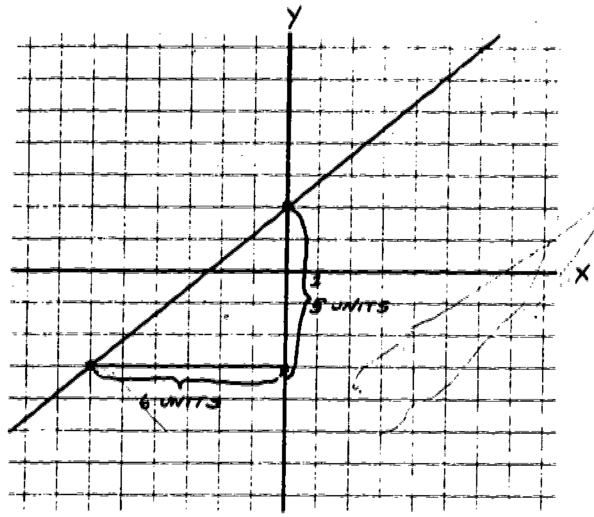
$x = -6$

80.



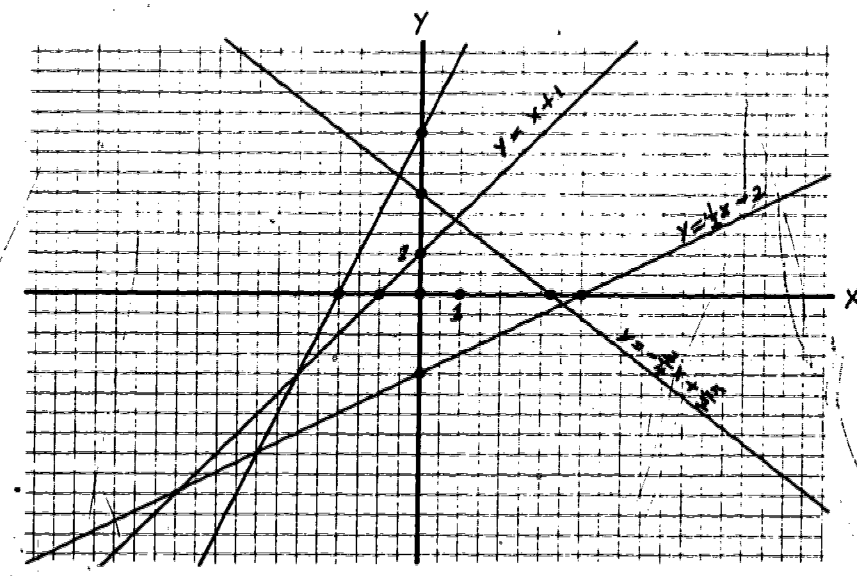
$y = -\frac{1}{2}x + 1$

81.



$y = \frac{5}{6}x + 2$

104 - 107



553

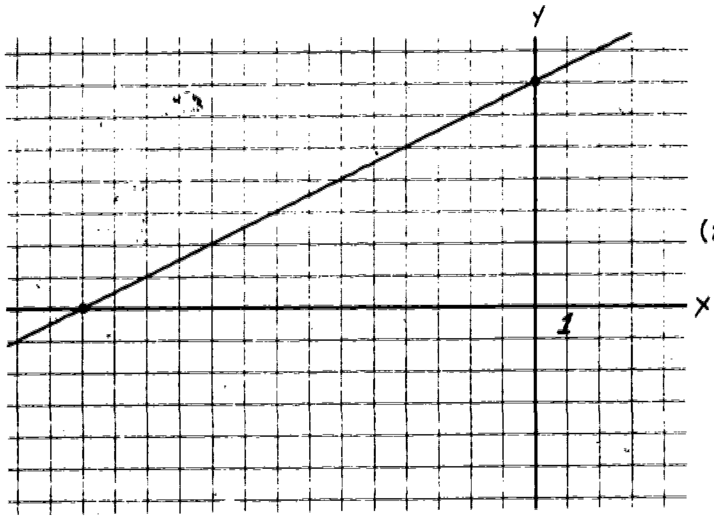
20-5. Summary and Review

1. (a) $y = 3x$

   (b) Since the line passes through the points $(0,-3)$ and $(-2,0)$, the slope is $\frac{3}{-2} = \frac{-3}{2}$ and the equation of the line is

$$y = \frac{-3}{2}x - 3. \quad (2y + 3x + 3 = 0).$$

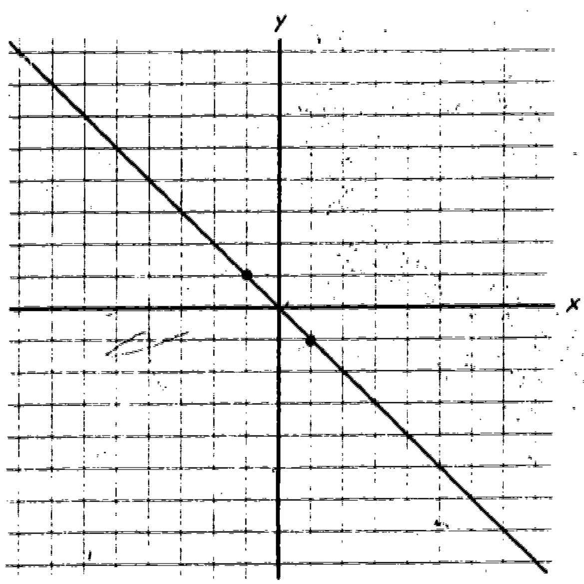
2. (a)



$y = \frac{x}{2} + 7$

$(2y - x - 7 = 0)$

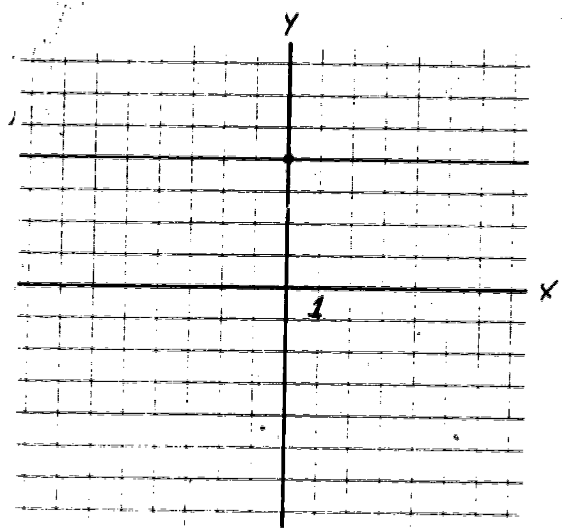
   (b)



$x + y = 0$

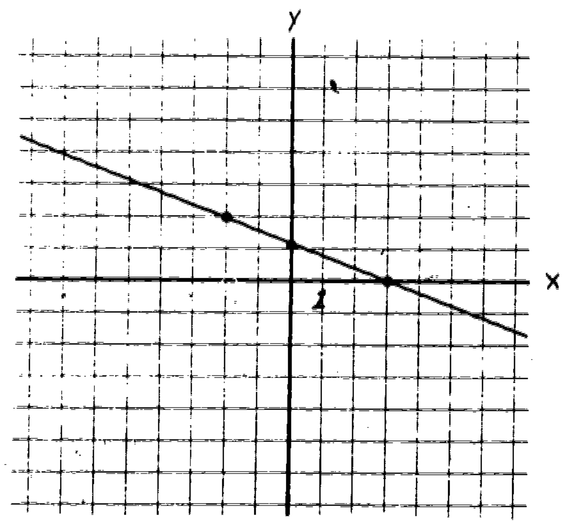
$(y = -x)$

2. (c)



$3y - 12 = 0$
$(y = 4)$

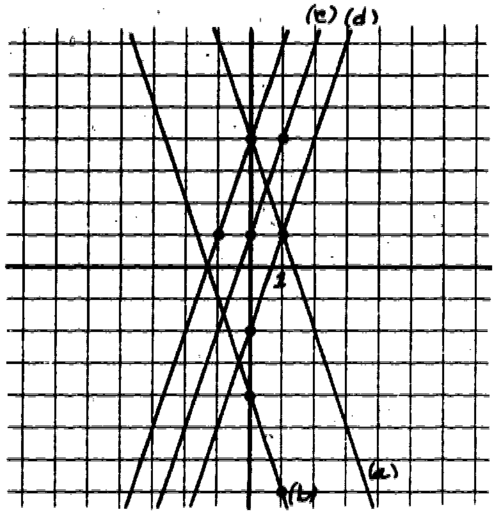
(d)



$2x + 5y - 6 = 0$
$(y = -\frac{2}{5}x + \frac{6}{5})$

3. (a) a is negative.
   (b) b is positive.
   (c) Coordinates of P: (a, -b)
   Coordinates of Q: (-a, -b)
   Coordinates of R: (-a, b)
   (d) If (c,d) is in the third quadrant,
   (c,-d) is in the second quadrant,
   (-c,d) is in the fourth quadrant,
   (-c,-d) is in the first quadrant.

4.

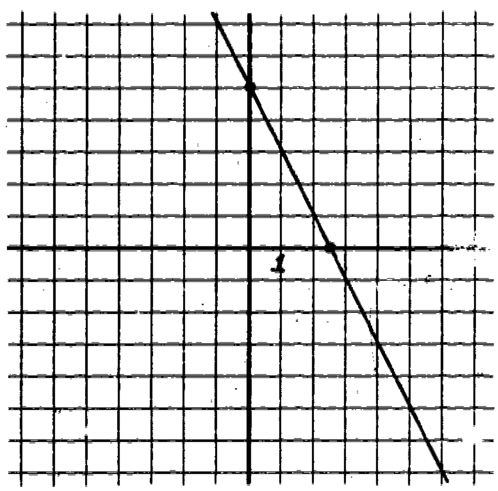


$y = 3x + 4$
$y = 3(-x) + 4$ (a)
$y = -(3x + 4)$ (b)
$y = (3x + 4) - 3$ (c)
$y = 3(x - 2) + 4$ (d)

Lines $y = 3x + 4$, $y = (3x + 4) - 3$ and $y = 3(x - 2) + 4$ are parallel lines.

Lines $y = 3(-x) + 4$ and $-(3x + 4)$ are parallel lines.

5. (a)

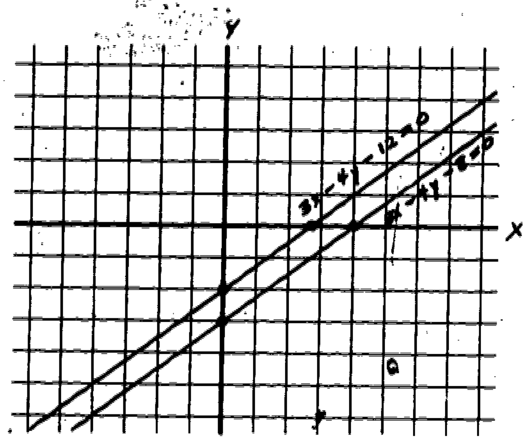


$2x + y - 5 = 0$ <u>also</u>
$6x + 3y - 15 = 0$

The equations $2x + y - 5 = 0$ and $6x + 3y - 15 = 0$ are <u>equivalent</u>. If we multiply both sides of the first equation by 3, we obtain the second equation: $3(2x + y - 5) = 3 \cdot 0$

$$6x + 3y - 15 = 0.$$

(b) If $k \neq 0$, $Ax + By + C = 0$ and $kAx + kBy + kC = 0$ are equivalent equations. Hence, they have the same truth sets and the graph is the same.

6. (a) 

The graphs of $3x - 4y - 12 = 0$ and $3x - 4y - 8 = 0$ are parallel lines. The coefficients of x and the coefficients of y are the same.

(b) If $k \neq 0$ and $D \neq kC$, the graphs of $Ax + By + C = 0$ and $kAx + kBy + D = 0$ are parallel.

If $B \neq 0$, the y-form of these equations are:

$$y = -\frac{A}{B}x - \frac{C}{B}$$

and

$$y = -\frac{A}{B}x - \frac{D}{kB}.$$

Note that the lines have the same slope.

7. (a) $y = 2x$.

(b) Slope is $\frac{9}{2}$, y-intercept is $(0,-6)$, equation is $y = \frac{9}{2}x - 6$.

(c) $y = x$.

(d) $y = -x$.

*8. (a) $(1,1)$ goes to $(-2,3)$
$(-1,-1)$ goes to $(-4,1)$
$(-2,2)$ goes to $(-5,4)$
$(0,-3)$ goes to $(-3,-1)$
$(3,0)$ goes to $(0,2)$.

(b) $(a, b-2)$ goes to $(2a - 3, 2b)$.

*(c) No point goes to itself. In order for a point to go to itself the sentences $a = a - 3$ and $b = b + 2$ must be true. However, the truth set of each of these sentences is the null set.

9. (a) $2w + 2(w + 3)$ or $4w + 6$.
This is linear in w.

(b) $w(w + 3)$ or $w^2 + 3w$.
This is not linear in w.

10. (a) πd. This is linear in d. If the diameter is doubled, the circumference is doubled. If the diameter is halved, the circumference is halved.

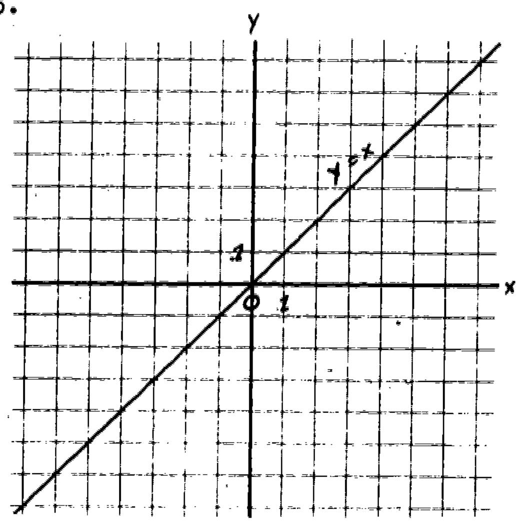
The ratio $\frac{c}{d}$ is constant and equal to the irrational number π.

The circumference varies directly with the diameter.

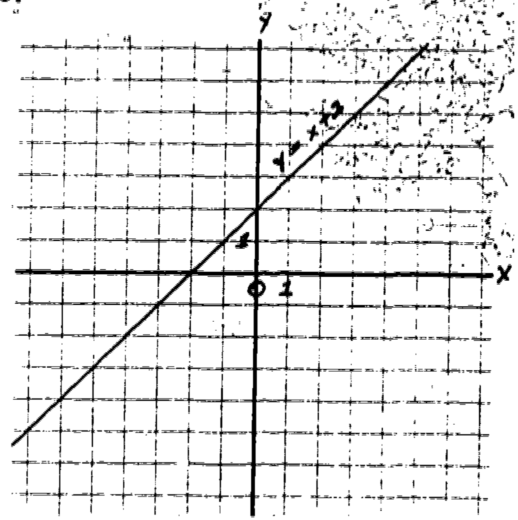
(b) $\frac{\pi d^2}{4}$. This is not linear in d. The ratio $\frac{A}{d}$ is not constant, $\frac{A}{d} = \frac{1}{4}\pi d$. The ratio $\frac{A}{d^2}$ is constant, $\frac{A}{d} = \frac{1}{4}\pi$.

21-1. Graphs of Inequalities

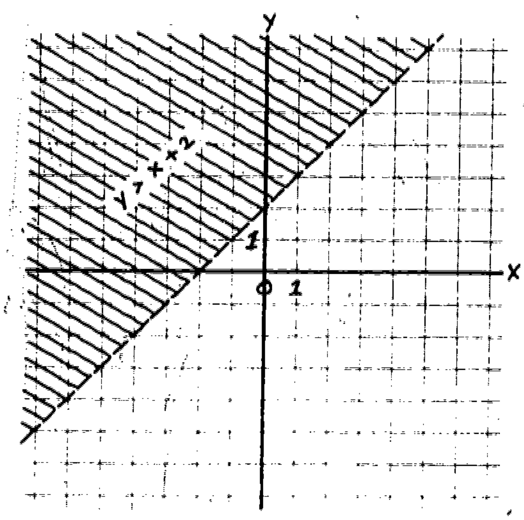
25.



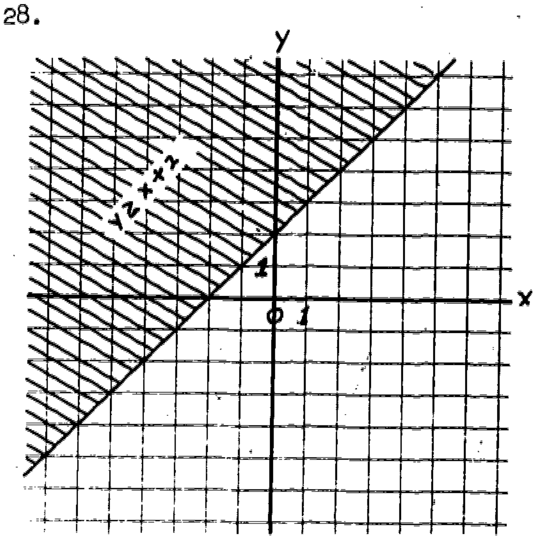
26.



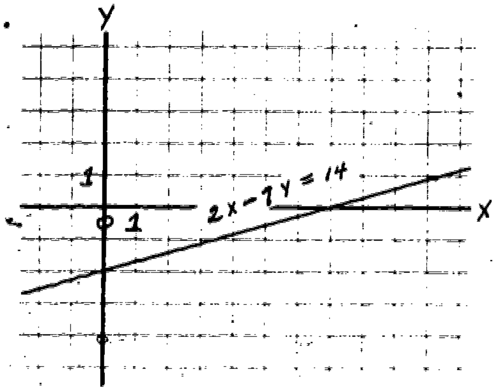
27.



28.



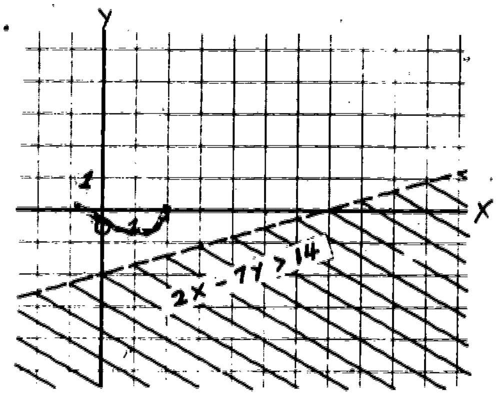
29.



$2x - 7y = 14$

$y = \frac{2}{7}x - 2$

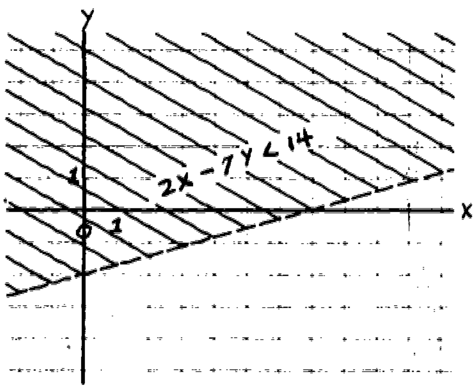
30.



$2x - 7y > 14$

$y < \frac{2}{7}x - 2$

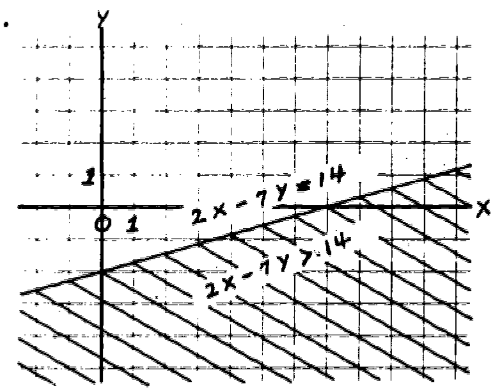
31.



$2x - 7y < 14$

$y > \frac{2}{7}x - 2$

32.

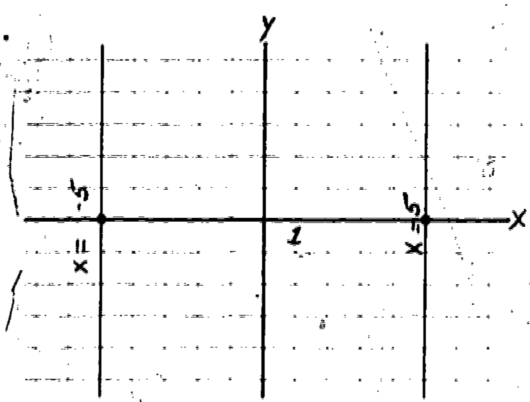


$2x - 7y \geq 14$

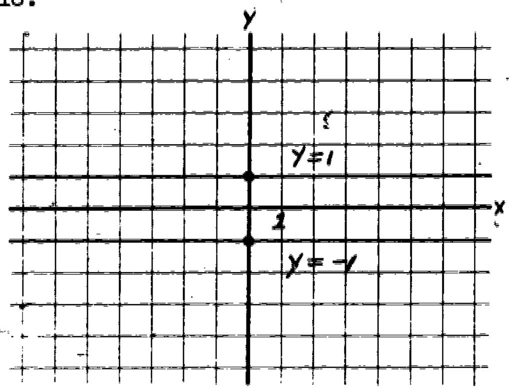
$y \leq \frac{2}{7}x - 2$

21-2. Graphs of Open Sentences Involving Absolute Value

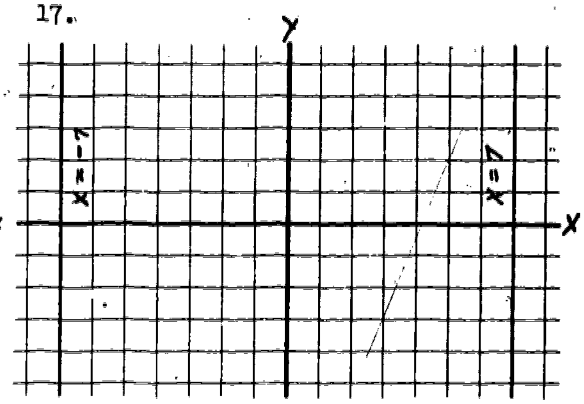
15.



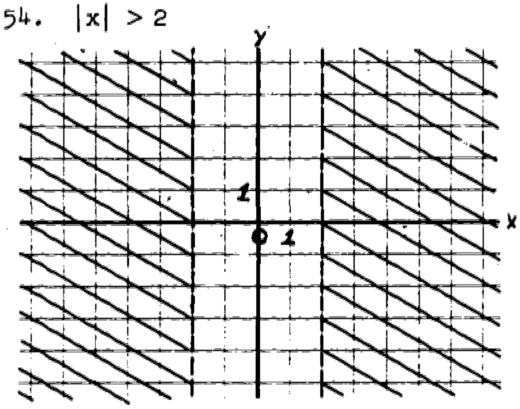
16.



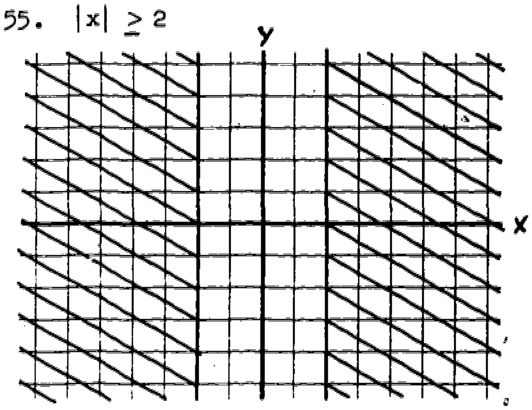
17.



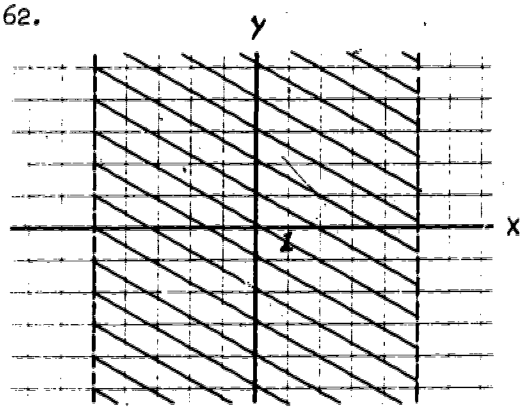
54. $|x| > 2$



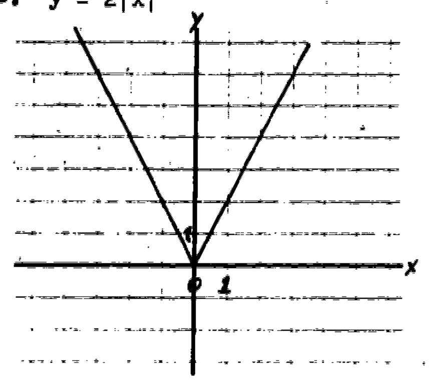
55. $|x| \geq 2$



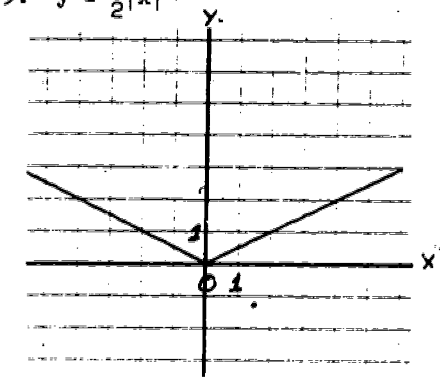
62.



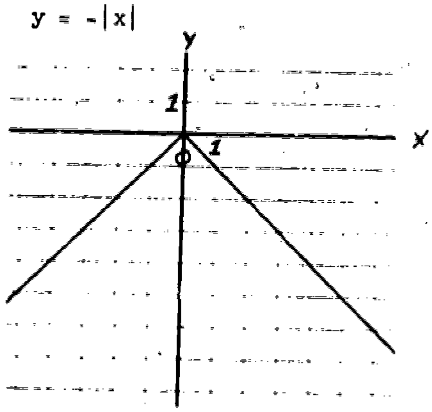
78. $y = 2|x|$



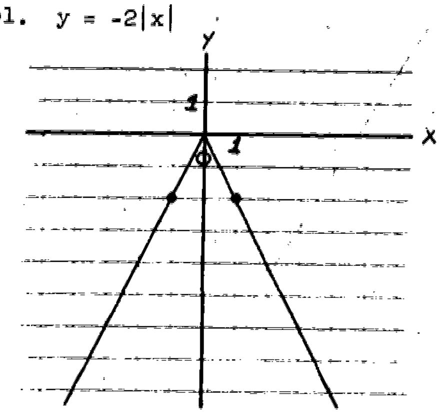
79. $y = \frac{1}{2}|x|$



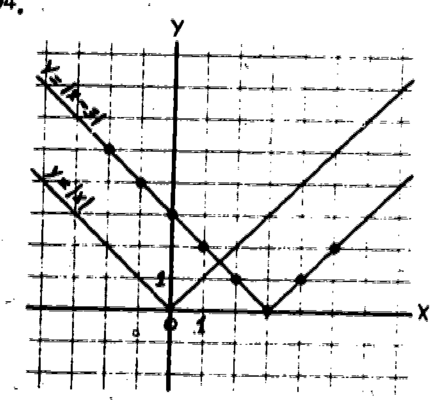
80. $y = -|x|$



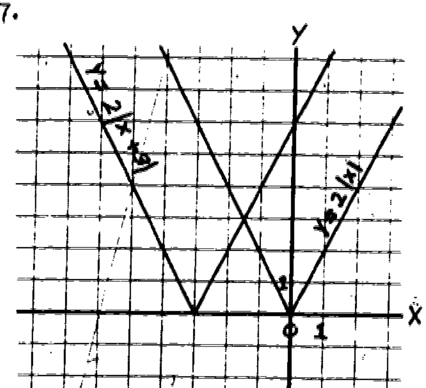
81. $y = -2|x|$



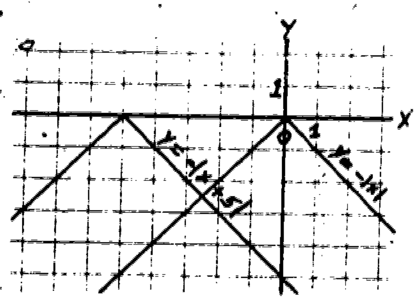
94.



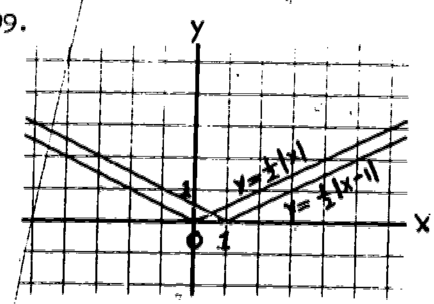
97.



98.



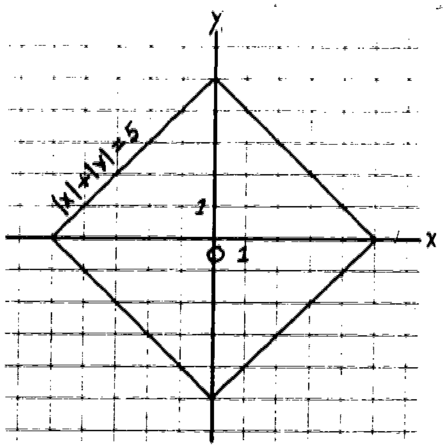
99.



*103 $|x| + |y| = 5$

x	-5	-3	-3	-1	-1	0	0	1	1	3	3	5
$\|x\|$	5	3	3	1	1	0	0	1	1	3	3	5
$\|y\|$	0	2	2	4	4	5	5	4	4	2	2	0
y	0	2	-2	4	-4	5	-5	4	-4	2	-2	0

*104.



The graph of $|x| + |y| = 5$ is given at the left.

Note that we could write four open sentences from which we could get the same graph:

$x + y = 5$ and $0 \leq x \leq 5$ or

$-x + y = 5$ and $-5 \leq x \leq 0$ or

$x - y = 5$ and $0 \leq x \leq 5$ or

$-x - y = 5$ and $-5 \leq x \leq 0$.

*21-3. <u>Graphs of Open Sentences Involving Integers Only</u>

23. $y = -3x + 1$, and x and y are integers.

24. $4 < x < 5$ and $-2 < y < 2$, and x and y are integers, or
 $5 \leq x \leq 7$ and $-1 \leq y \leq 1$, and x and y are integers.

25. $y = -x - 3$ for $-8 < x < -4$, and x and y are integers.

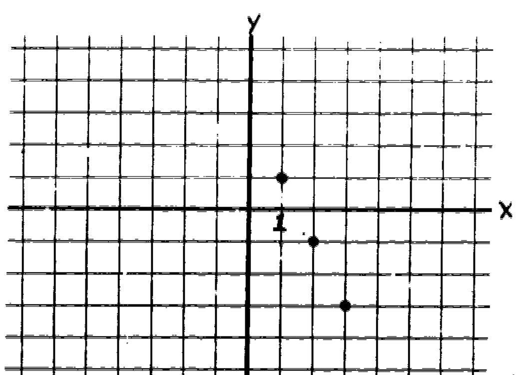
26. $x = -2$ and $2 < y < 7$, and x and y are integers.

27. $-2 < x < 2$ and $-4 < y < 3$, and x and y are integers.

28. $x > -6$ and $y < 6$ and $y \geq x + 6$, and x and y are integers.

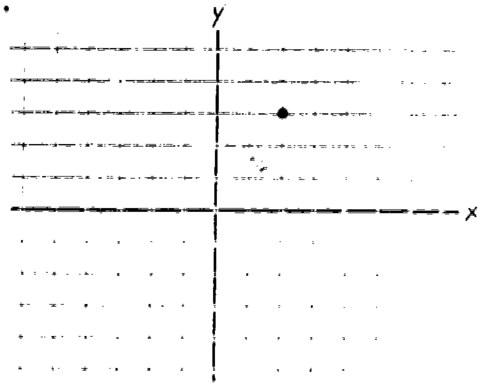
29. $x = -4$ or $-7 < y < -2$, and x and y are integers.

30.



$y = -2x + 3$ and $1 \leq x \leq 3$, and x and y are integers.

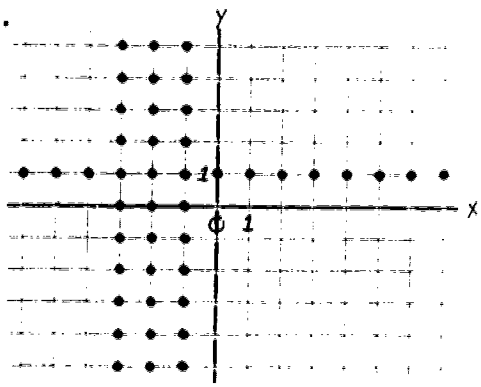
31.



$y > \frac{[illegible]}{2}x - 1$ and $x > 0$ and $y < [illegible]$, and x and y are integers.

Note that [illegible] is the only value of x that will allow y to be an integer [illegible]. If x [illegible], then y is [illegible].

32.



$-3 \leq x \leq -1$ <u>or</u> $0 < y < [illegible]$, and x and y are integers.

This may also be written as $-4 < x < -1$ <u>or</u> $y = 1$ and x and y are integers. The set of points is infinite.

21-4. Summary and Review

1. (a) $y = |x|$

   (b) $|x| > 5$, or $x > 5$ or $x < -5$

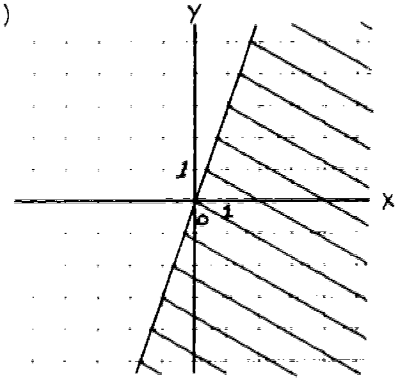
   (c) $y = |2x - 4| = 2|x - 2|$

   (d) $y \leq \frac{3}{2}x + 2$

   (e) $y \geq 2|x - 3|$

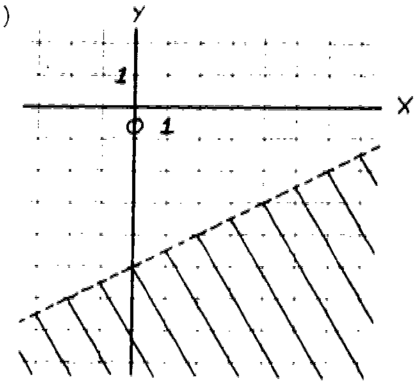
   (f) $y > -x + 4$

2. (a)



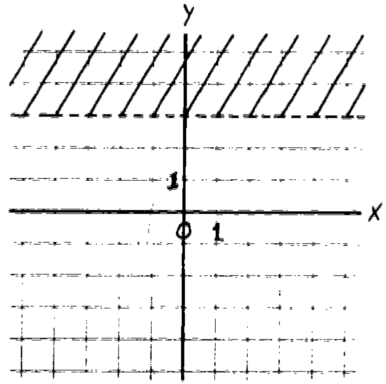
$y \leq 3x$

(b)



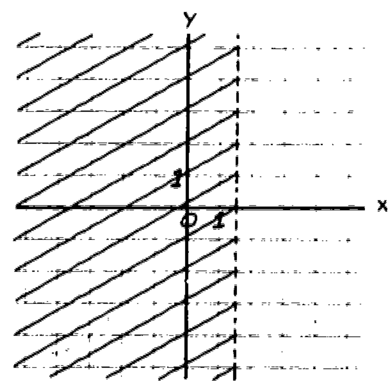
$y < \frac{x}{2} - 5$

(c)



$y > 3$

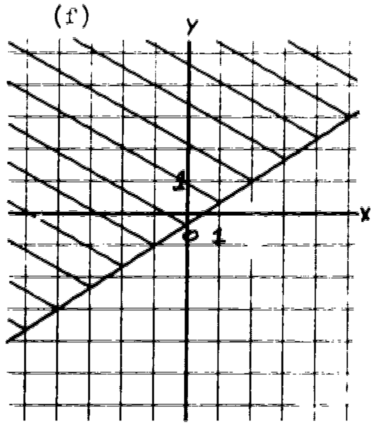
(d)



$x < 1.5$

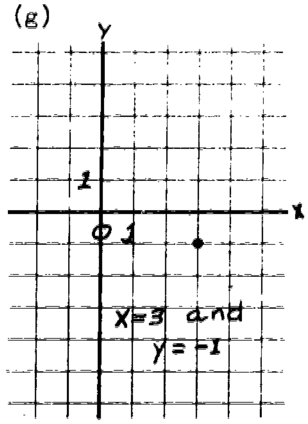
*(e) The graph is not possible since "$|x| + |y|$" <u>must</u> be positive. Therefore, the truth set of "$|x| + |y| = -2$" <u>is</u> $\emptyset$.

(f)

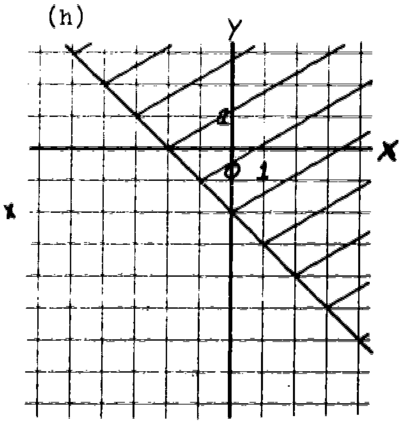


$3y \geq 2x - 1$

(g)

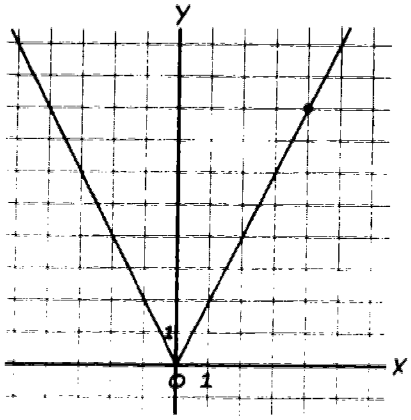


(h)



$x + y \leq -2$

595

3. 

$y = 2|x|$

(a) $y = -2|x|$

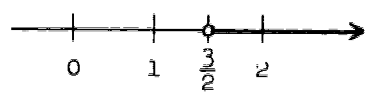
(b) $y = 2|x - 3|$

(c) $y = 2|x + 2|$

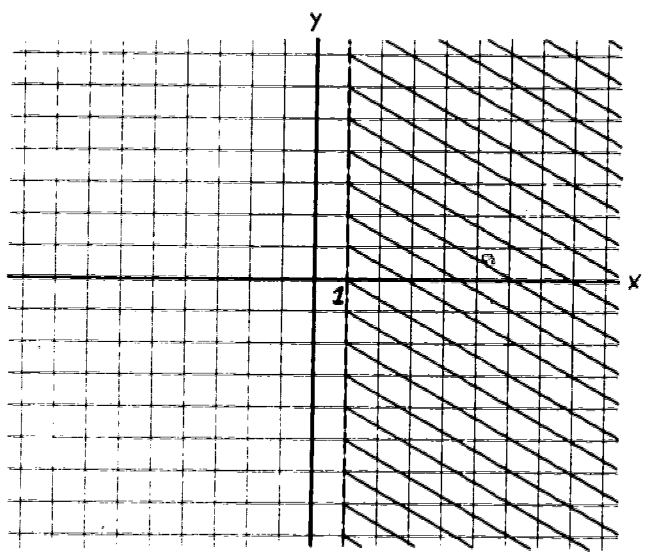
(d) $y = 2|x| + 5$

(e) $y = 2|x - 2| - 4$

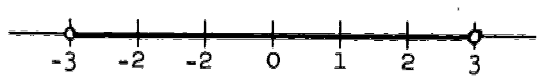
4. (a) $2x - 3 > 0$. Graph of $2x - 3 > 0$ in one variable.



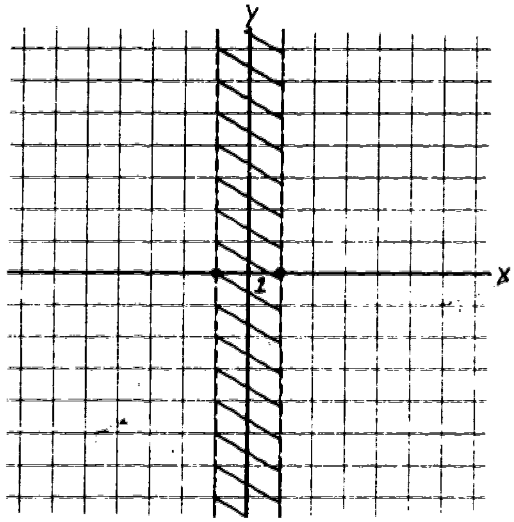
(b) Graph of $2x - 3 > 0$ in two variables:



5. (a) $|y| < 3$. Graph of $|y| < 3$ in one variable:



5. (b)



*6. (a) $x < 6$ and $y > 0$ and $y \leq x$ and x and y are integers, or
$x \leq 5$ and $y \geq 1$ and $y \leq x$ and x and y are integers.

(b) $|x| + |y| \leq 8$.

7. If the two-digit number is $10t + u$, the sum of its digits is $t + u$. An open sentence for the problem is

$$\frac{10t + u}{t + u} = 4 + \frac{3}{t + u}.$$

Then
$$10t + u = 4t + 4u + 3$$
$$6t - 3u = 3$$
$$2t - u = 1$$
$$u = 2t - 1, \quad 0 < u \leq 9.$$

If $t = 1$, then $u = 1$.
If $t = 2$, then $u = 3$.
If $t = 3$, then $u = 5$.
If $t = 4$, then $u = 7$.
If $t = 5$, then $u = 9$.

If $t = 1$, $u = 1$, then $\frac{10t + u}{t + u}$ is $\frac{11}{2} = 4 + \frac{3}{2}$. The pair of values, $t = 1$, $u = 1$, are not allowable since the remainder 3 is greater than the divisor 2 and this is not possible.

If $t = 2$, and $u = 3$, $\frac{23}{5} = 4 + \frac{3}{5}$.

If $t = 3$, and $u = 5$, $\frac{35}{8} = 4 + \frac{3}{8}$.

If $t = 4$, and $u = 7$, $\frac{47}{11} = 4 + \frac{3}{11}$.

If $t = 5$, and $u = 9$, $\frac{59}{14} = 4 + \frac{3}{14}$.

The possible solutions are 23, 35, 47, 59.

*8. If the number of steers is s and number of cows is c, then the open sentence is

$$
\begin{aligned}
25s + 26c &= 1000 \\
25s &= 1000 - 26c \\
s &= \frac{1000 - 26c}{25} \\
s &= 40 - \frac{26c}{25}
\end{aligned}
$$

If s and c are positive integers, then $26c$ must be divisible by 25. This is true when $c = 25, 50, 75, \ldots,$ a multiple of 25, since 26 and 25 have no common factors greater than 1.

If $c = 25$, $\frac{26c}{25} = 26$ and $s = 40 - 26 = 14$.

If $c = 50$, $\frac{26c}{25} = 52$ and $s = 40 - 52 = -12$.

If $c = 75$, $\frac{26c}{25} = 78$ and $s = 40 - 78 = -38$.

It is thus apparent that if $c \geq 50$, s is a negative number. Hence, c may only be 25

$$
\begin{aligned}
\text{and}\quad s &= 40 - 26 \\
s &= 14.
\end{aligned}
$$

So he may buy 25 cows and 14 steers.

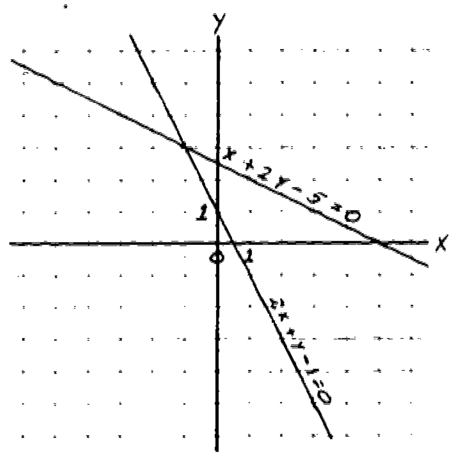
If we were to solve the original equation instead for c,

$$c = \frac{1000 - 25s}{26}$$

s would have to be chosen so as to make $1000 - 25s$ divisible by 26. Though this can be done, it is plainly more difficult than the other approach.

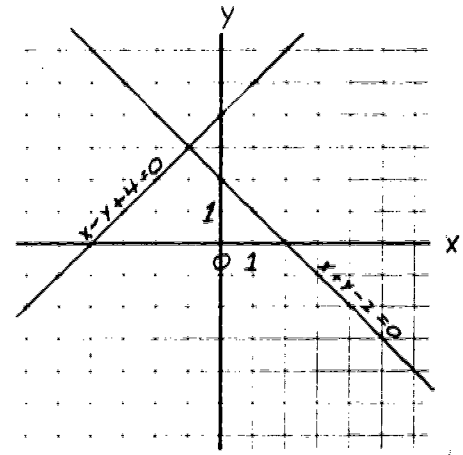
22-1. Systems of Equations

5.



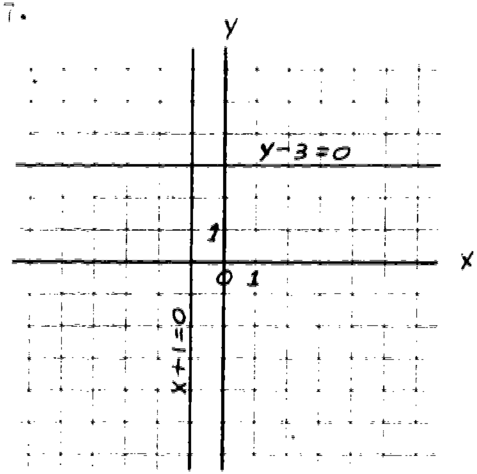
$x + 2y - 5 = 0$ or $2x + y - 1 = 0$

6.



$x + y - 2 = 0$ or $x - y + 4 = 0$

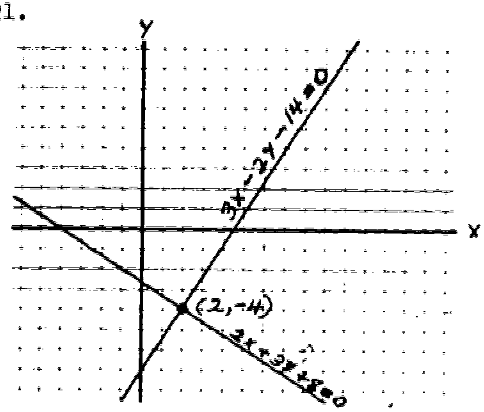
7.



$x + 1 = 0$ or $y - 3 = 0$

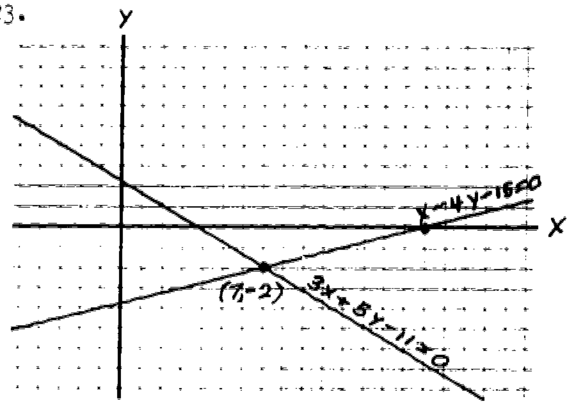
539

21.



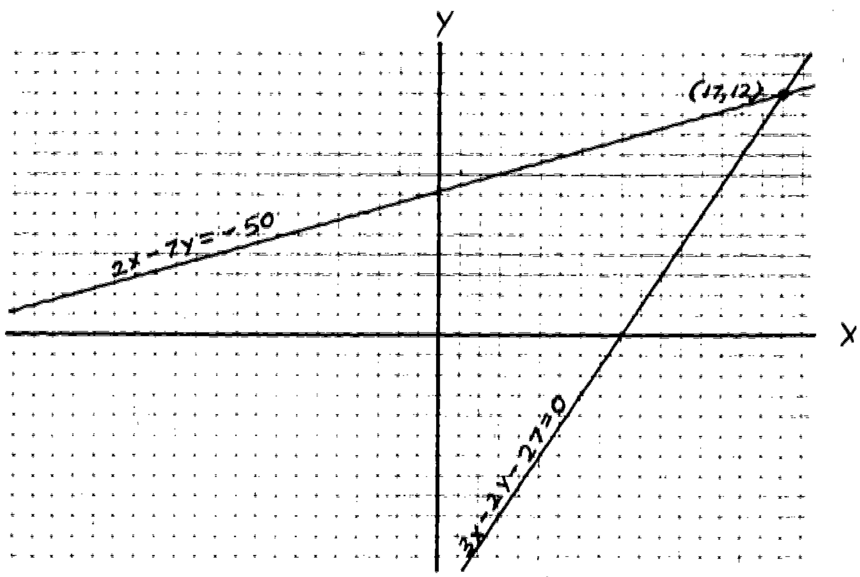
$$\begin{cases} 3(2) - 2(-4) - 14 = 0 \\ 2(2) + 3(-4) + 8 = 0 \end{cases}$$

23.



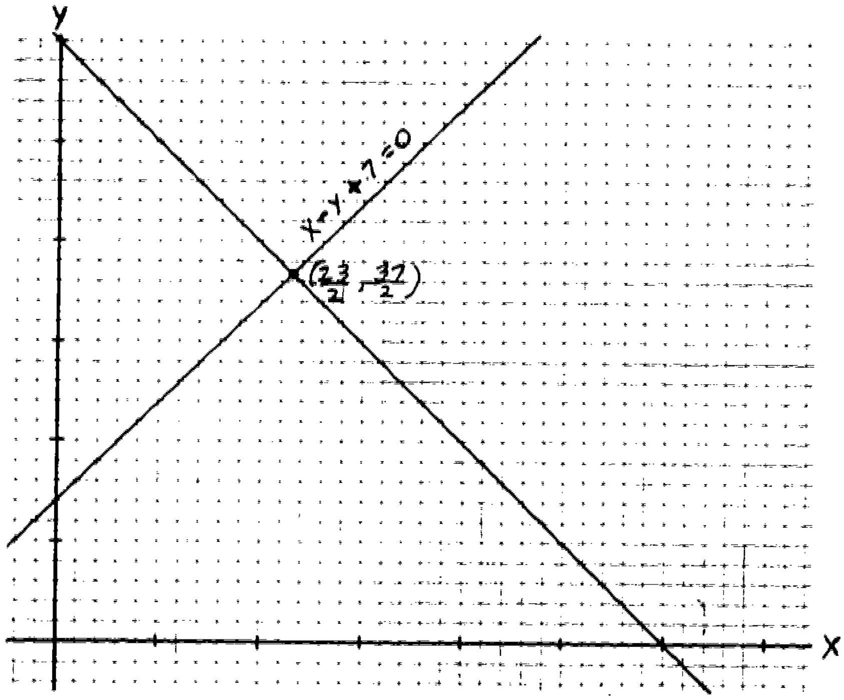
$$\begin{cases} (7) - 4(-2) - 15 = 0 \\ 3(7) + 5(-2) - 11 = 0 \end{cases}$$

22.



$$\begin{cases} 2(17) - 7(12) = -50 \\ 3(17) - 2(12) - 27 = 0 \end{cases}$$

24.



$$\begin{cases} \frac{23}{2} + \frac{37}{2} - 30 = 0 \\ \frac{23}{2} - \frac{37}{2} + 7 = 0 \end{cases}$$

22-2. Systems of Equations (Continued)

Note: Answers for Items *33-*36 may have r and s interchanged.

*34. $(r + s)x + (r - s)y - 2r + 4s = 0$

*35. $(5r + s)x + (-r + s)y - 10r - 8s = 0$

*36. $(3r + 2s)x + (-2r + 3s)y - 14r + 8s = 0$

*37. $(3r - 6s)x + (12r + 15s)y - 5r + 4s = 0$

*62. Having discovered that we can use $r = 6$, $s = 4$, we note that 2 is the greatest common factor of 6 and 4. That is, $r = 2 \cdot 3$, $s = 2 \cdot 2$. If $r = 6$, $s = 4$ makes the coefficient of x equal to 0, then $r = 3$, $s = 2$ will accomplish the same. This is justified by noting that from

$$r(4x + 21y - 27) + s(-6x + 15y - 37) = 0$$

we have

$$(4r - 6s)x + (21r + 15s)y - 27r - 37s = 0.$$

If $4r - 6s = 0$, $4r = 6s$

$$r = \frac{6s}{4} = \frac{3s}{2}.$$

81. $\begin{cases} y = \frac{2}{3}x + 2 \\ y = -\frac{5}{2}x + 40 \end{cases}$

$$\frac{2}{3}x + 2 = -\frac{5}{2}x + 40$$
$$4x + 12 = -15x + 240$$
$$19x = 228$$
$$x = 12$$

When $x = 12$
$$y = \frac{2}{3}(12) + 2$$
$$y = 10.$$

The solution set is $\{(12,10)\}$.

Check: $10 = \frac{2}{3}(12) + 2$
$$10 = 8 + 2$$
$$10 = 10$$

and $10 = -\frac{5}{2}(12) + 40$
$$10 = -30 + 40$$
$$10 = 10$$

83. $\begin{cases} x = \frac{3}{2}y - 4 \\ y = -\frac{2}{3}x \end{cases}$

$$\begin{cases} 2x = 3y - 8 \\ 3y = -2x \end{cases}$$
$$2x = -2x - 8$$
$$4x = -8$$
$$x = -2$$

when $x = -2$
$$y = -\frac{2}{3}(-2)$$
$$y = \frac{4}{3}.$$

The solution set is $\{(-2, \frac{4}{3})\}$.

Check: $-2 = \frac{3}{2}(\frac{4}{3}) - 4$
$$-2 = 2 - 4$$
$$-2 = -2$$

and $\frac{4}{3} = -\frac{2}{3}(-2)$
$$\frac{4}{3} = \frac{4}{3}.$$

86. $\begin{cases} \frac{x}{2} - \frac{x}{3} = 1 \\ x + y = 7 \end{cases}$

$$6\cdot\frac{x}{2} - 6\cdot\frac{x}{3} = 6\cdot 1$$
$$3x - 2x = 6$$
$$x = 6$$

when $x = 6$
$$6 + y = 7$$
$$y = 1$$

The solution set is $\{(6,1)\}$.

Check: $\frac{6}{2} - \frac{6}{3} = 1$
$$3 - 2 = 1$$
$$1 = 1$$

and $6 + 1 = 7$
$$7 = 7.$$

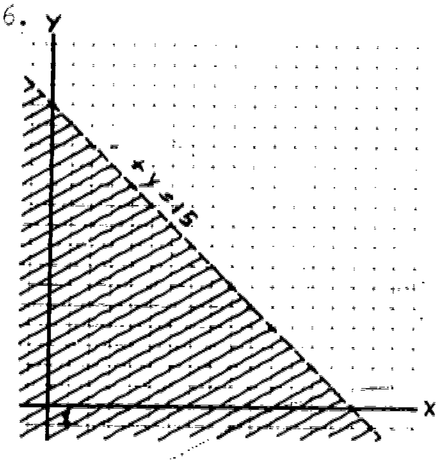
88. $\begin{cases} 6y + (2 - 4x) = 3 \\ 4x - 2(3y - 1) = 2 \end{cases}$

$$\begin{cases} -4x + 6y - 1 = 0 \\ 4x - 6y = 0 \end{cases}$$

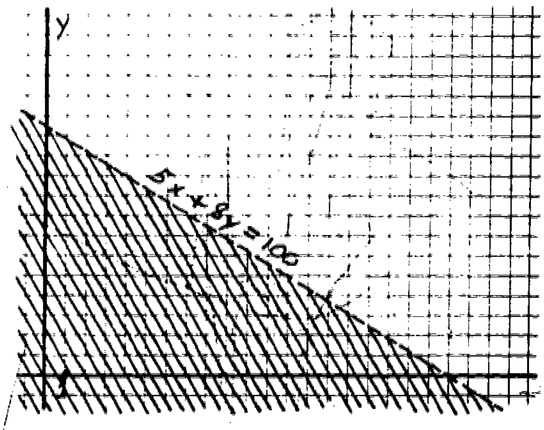
The graphs of the open sentences are parallel lines. The slopes are the same. The y-intercepts are different. The solution set is $\emptyset$.

22-4. Systems of Inequalities

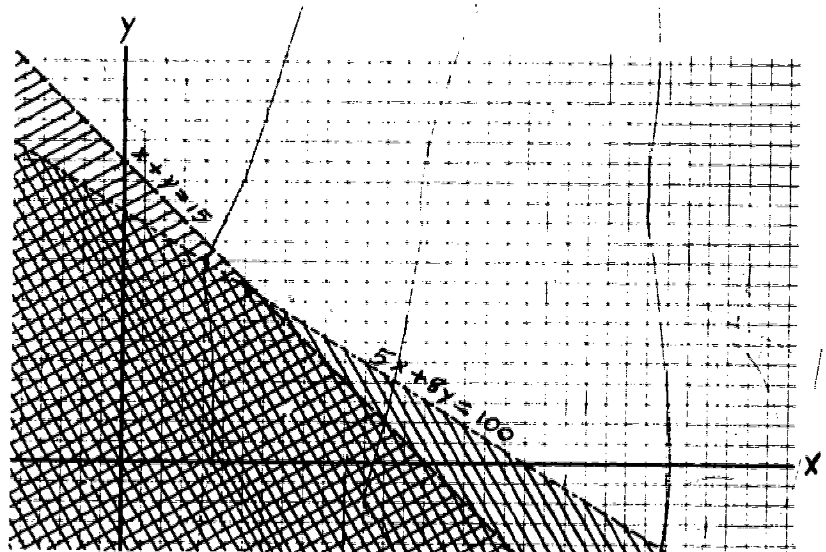
6.



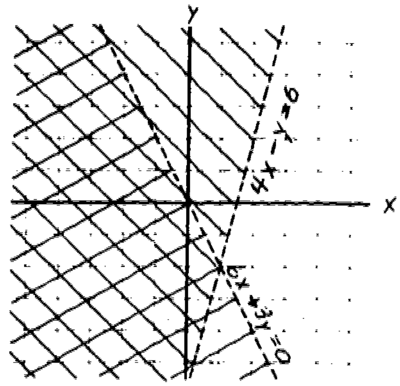
7.



8.

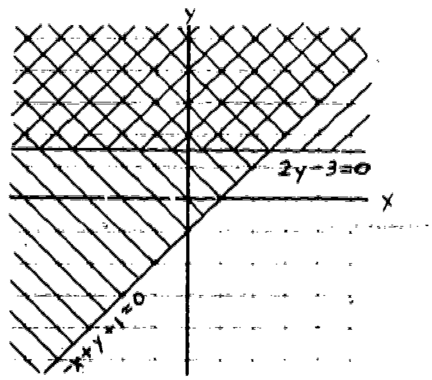


11.



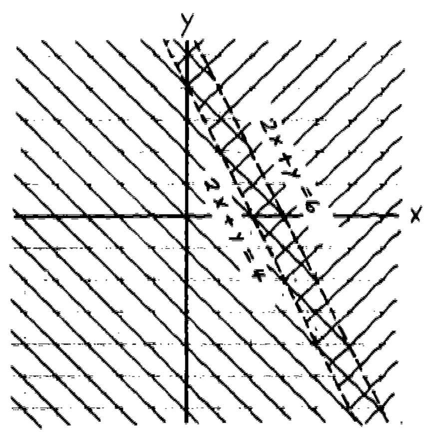
Graph is doubly shaded region.

12.



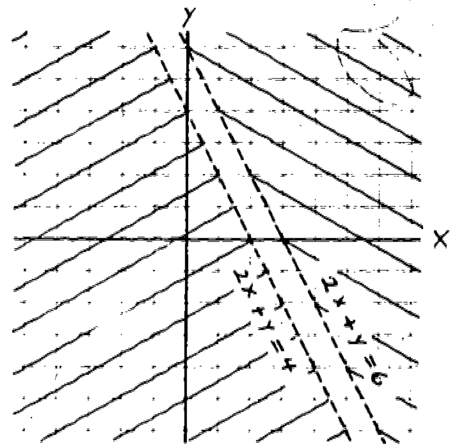
Graph consists of doubly shaded region, plus both boundaries.

13.



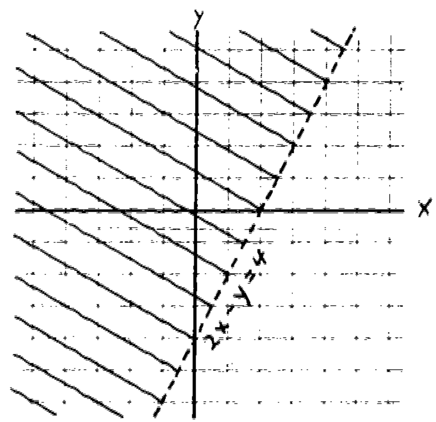
Graph is doubly shaded region.

14.



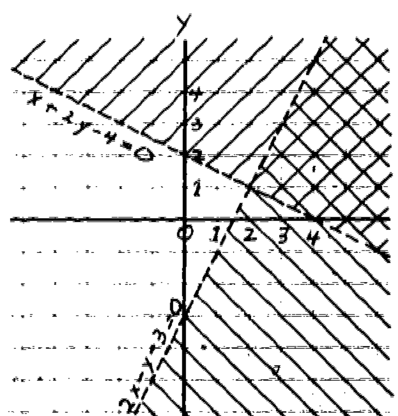
The truth set of the system is $\emptyset$.

15.

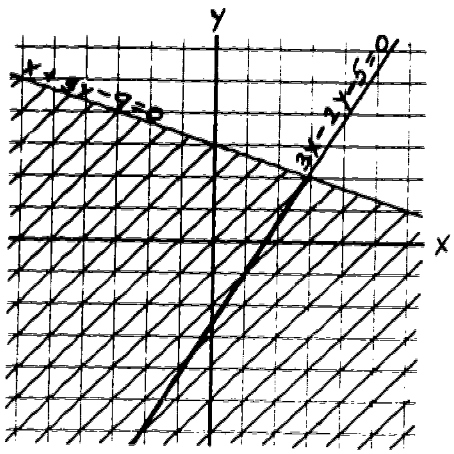


$4x - 2y < 8$ is equivalent to $2x - y < 4$; the intersection of this with $2x - y \leq 4$ is $2x - y < 4$. The graph of the system does not include the boundary.

16.

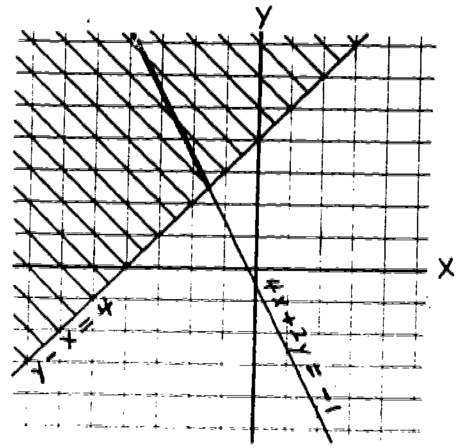


21.



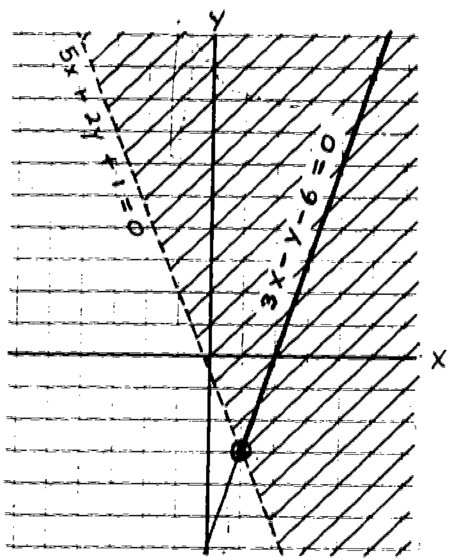
Graph consists of heavy portion of the line $3x - 2y - 5 = 0$.

22.



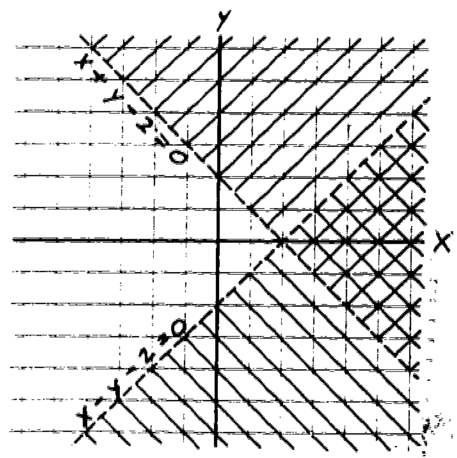
Graph consists of heavy portion of the line.

23.

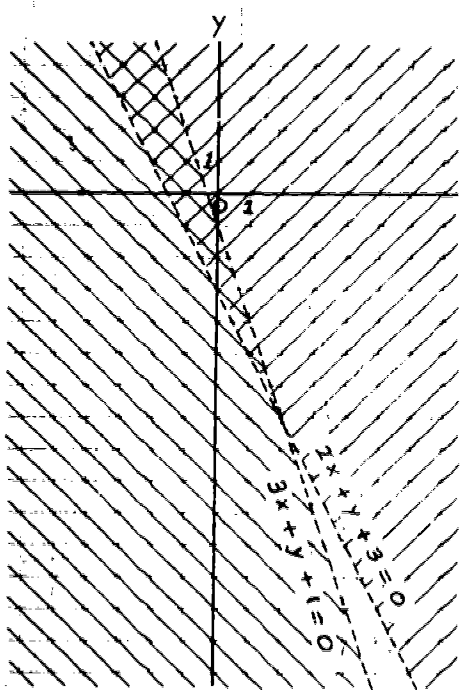


Graph consists of heavy portion of the line. The circle around the point (1,-3) indicates that this point is not included in the graph of the truth set.

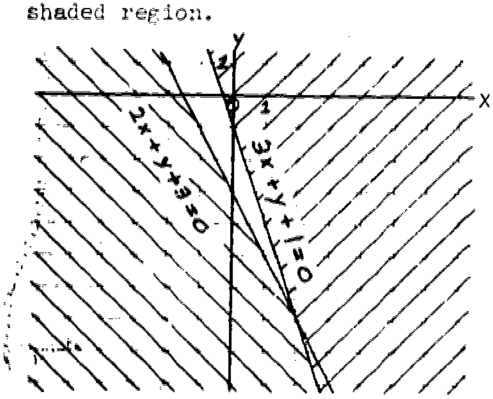
24.



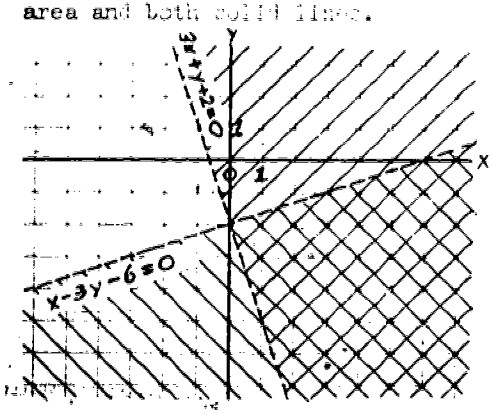
The graph consists of the entire shaded region.



The graph consists of the entire shaded region.

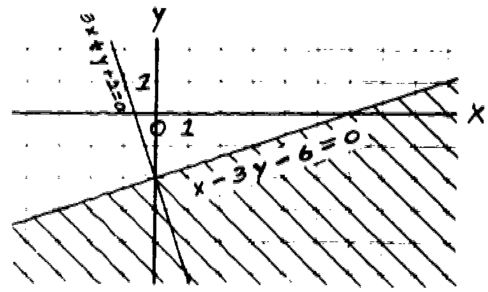


Graph consists of entire shaded area and both solid lines.

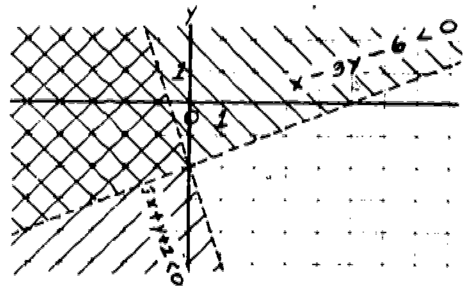


The graph is the entire shaded region.

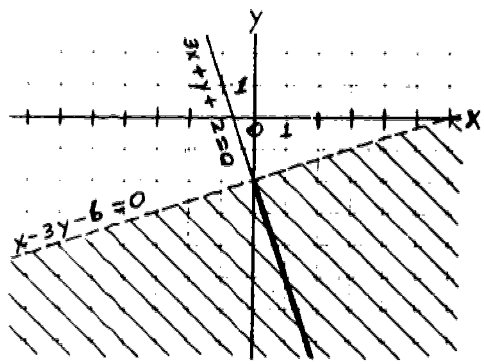
28.



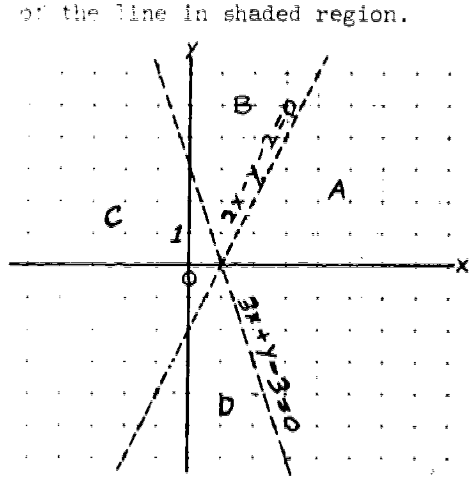
The graph is the entire shaded region plus the lines.



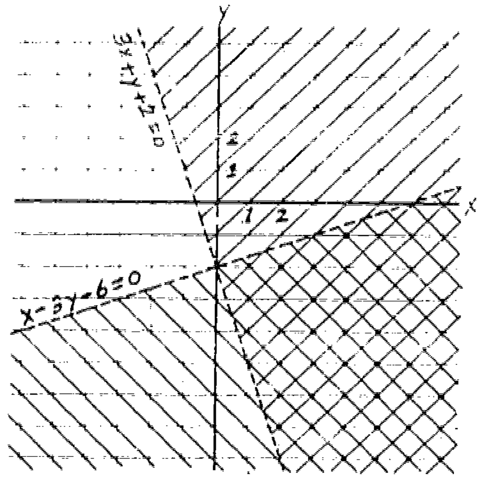
The graph is the double shaded area.



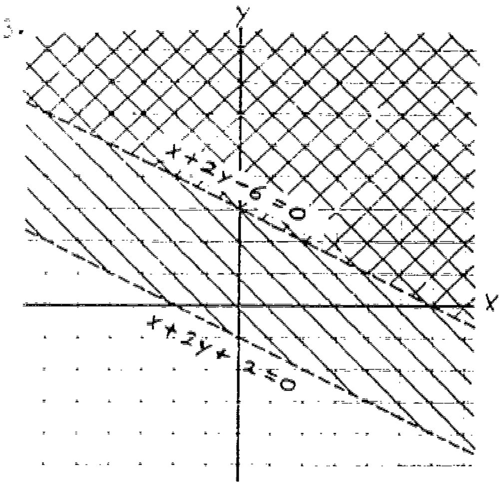
The graph is the heavy portion of the line in shaded region.



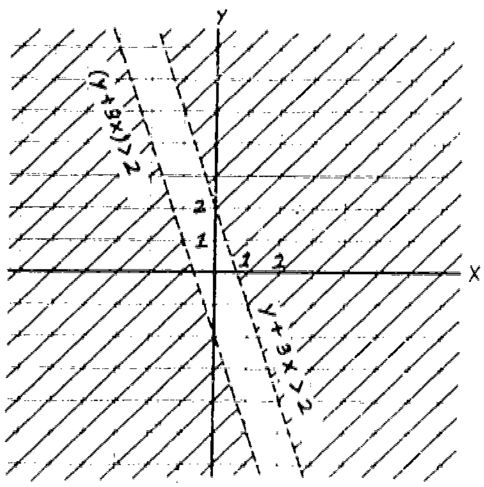
61.



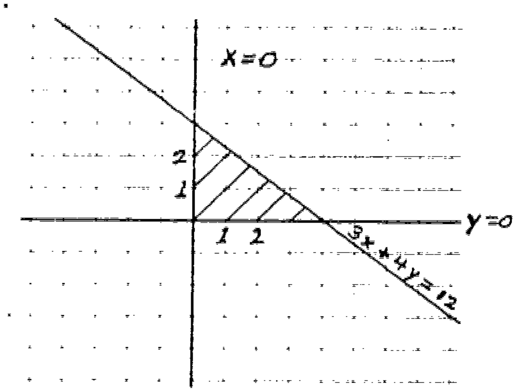
63.



68.

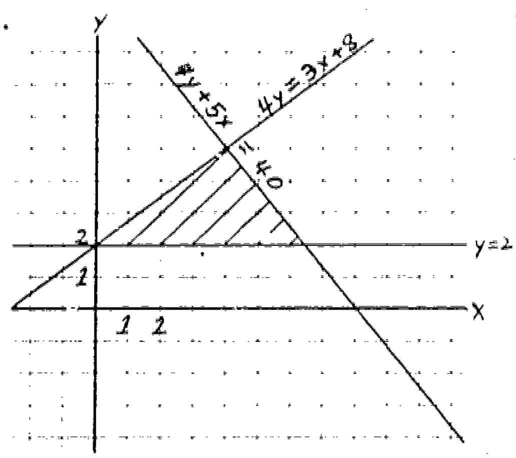


69.



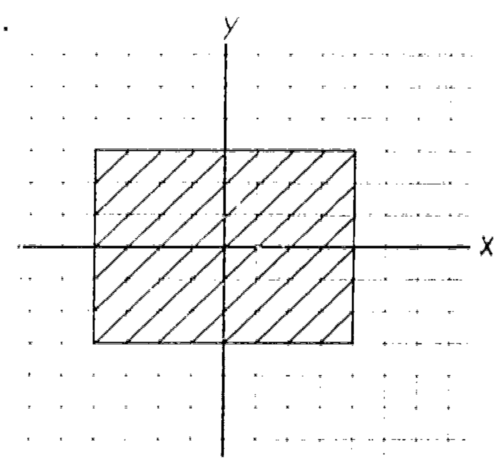
The graph is the shaded region including all boundaries.

70.



The graph is the shaded region, including all three boundaries.

72.



The graph is the shaded region including all of the boundaries.

74. This example is illustrative of some problems in the field of linear programming. This field is becoming increasingly important in the applications of mathematics. Corresponding to the geometric aspects, the boundaries give us what is referred to as a convex set of points.

If r is the number of running plays and p is the number of pass plays, then $3r$ is the number of yards made on r running plays and $20(\frac{1}{3})p$ is the number of yards made on p passing plays. Since the team is 60 yards from the goal line,

$$3r + \frac{20}{3}p \geq 60$$

if they are to score.

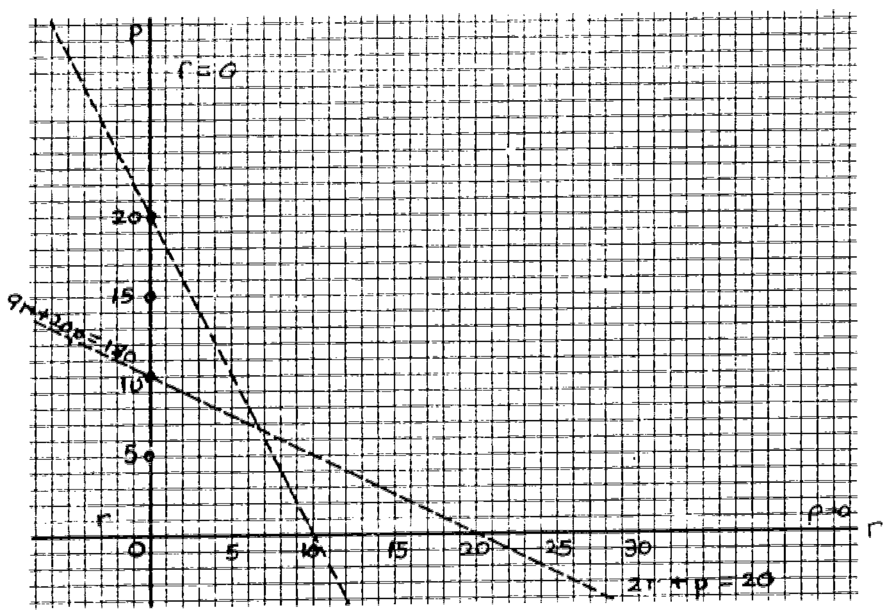
$30r$ seconds are required for r running plays, and $15p$ seconds are required for p passing plays; therefore,

$$30r + 15p \leq 5(60).$$

These two inequalities give the equivalent system

$$\begin{cases} 20p + 9r \geq 180 \\ p + 2r \leq 20 \end{cases} \quad \text{(}p \text{ and } r \text{ are non-negative integers.)}$$

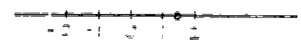
The graph of this system is



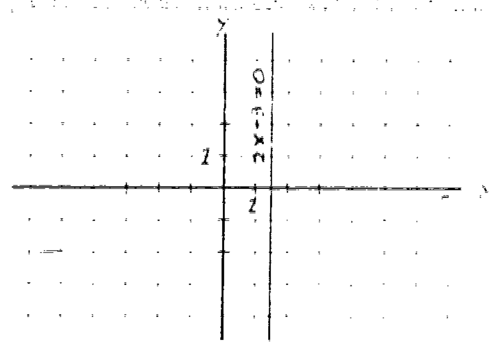
There are 48 different combinations of r and p which will assure success; for example, 4 running plays and 8 passes. However, some combinations, such as 6 runs, 7 passes, leave less time remaining, thus giving the opponents less time to try to score. These combinations correspond to points of the graph nearest to the line $p + 2r = 20$.

22-5. Review

1. (a) As an equation in one variable, the truth set of "$2x - 3 = 0$" is $\frac{3}{2}$ and its graph is:

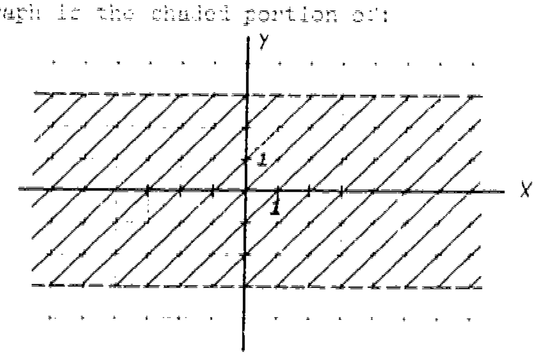


(b) As an equation in two variables, its truth set is the set of all ordered pairs of real numbers with first number $\frac{3}{2}$. Its graph is:



2. (a) As a sentence in one variable, the truth set of "$|y| < 3$" is the set of all y such that $-3 < y < 3$, and its graph is:

(b) As a sentence in two variables, its truth set is the set of all ordered pairs of real numbers with second number between -3 and 3. The graph is the shaded portion of:



3. The line $r(3x - 5y - 4) + s(2x + 3y + 4) = 0$ contains the point of intersection for any numbers r and s, not both 0. Let $r = -2$ and $s = 3$. Then

$$-2(3x - 5y - 4) + 3(2x + 3y + 4) = 0$$
$$19y + 20 = 0$$

is the equation of the required horizontal line. Let $r = 3$ and $s = 5$. Then

$$3(3x - 5y - 4) + 5(2x + 3y + 4) = 0$$
$$19x + 8 = 0$$

is the equation of the required vertical line.

4. (a) $\{(5,3)\}$ [illegible]

(b) $\{(-2,-1)\}$

(c) Set of all solutions of either equation.

(d) $\emptyset$

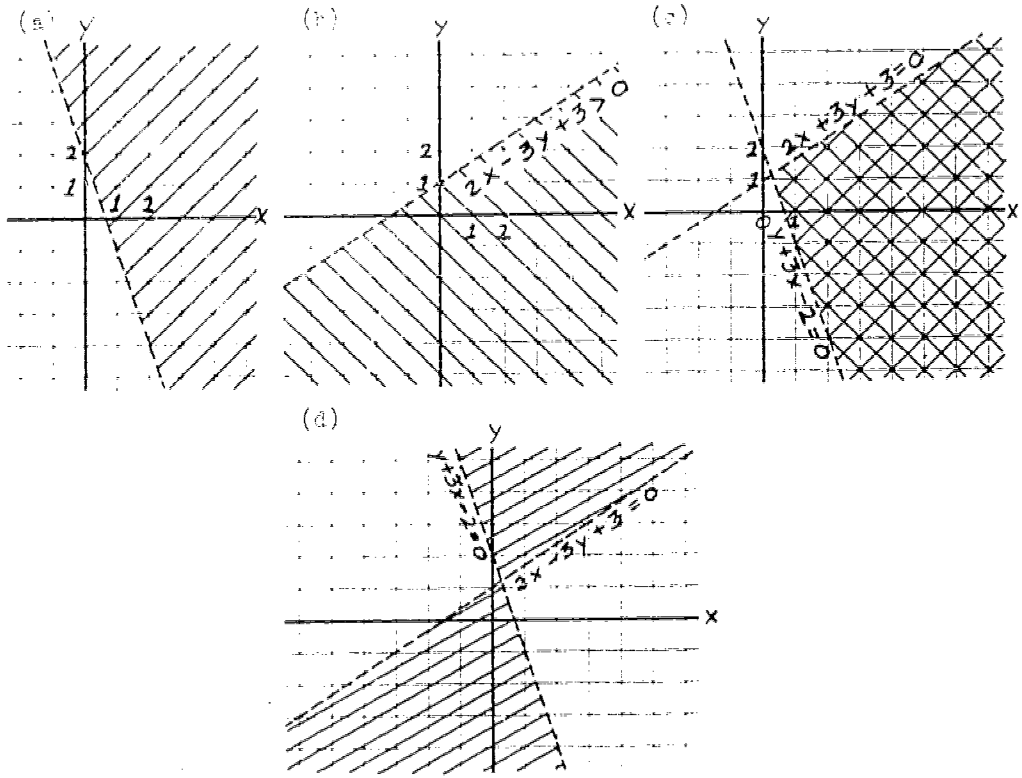
(e) [illegible]

(f) [illegible]

5. (a) The lines $Ax + By + C = 0$ and $Dx + Ey + F = 0$ are parallel if and only if either $B = E = 0$, and $\frac{A}{D} \neq \frac{C}{F}$ or $\frac{A}{D} = \frac{B}{E} \neq \frac{C}{F}$.

(b) If $\frac{A}{D} = \frac{B}{E} = \frac{C}{F}$ the equations represent the same non-vertical line.

(c) The lines have exactly one common point if and only if both have slopes and their slopes are different, or if one has a slope and the other does not.

6. 

7. (a) $\begin{cases} y = x + 1 \\ x + y = 57, \end{cases}$ x and y integers.

Truth set: $\{(28,29)\}$

(b) $\begin{cases} x + y = 16 \\ 2x = y - 3, \end{cases}$ x and y integers.

Truth set: $\emptyset$

(c) $\begin{cases} x + y = 45 \\ y = 4x + 5 \end{cases}$

Truth set: $\{(8,37)\}$

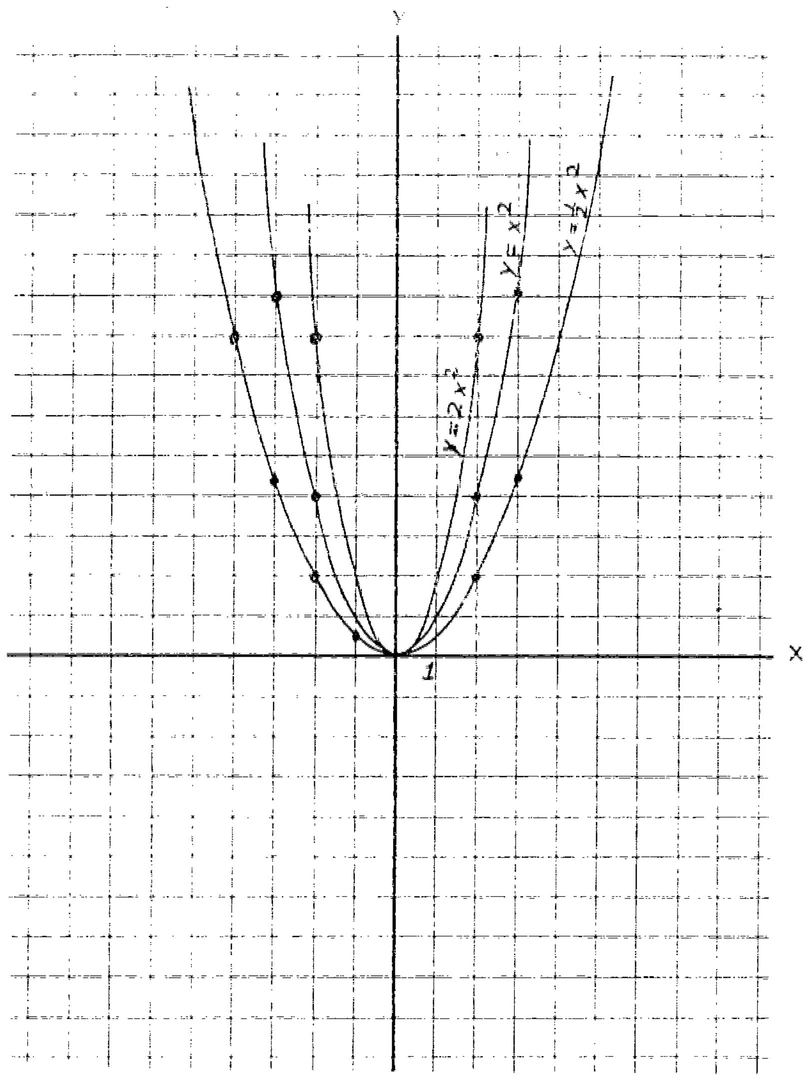
(d) $\begin{cases} x + y = 20 \\ 4.8x + 6.0y = 110. \end{cases}$

Truth set: $\{(8\frac{1}{3}, 11\frac{2}{3})\}$

Chapter 23 - GRAPHS OF QUADRATIC POLYNOMIALS

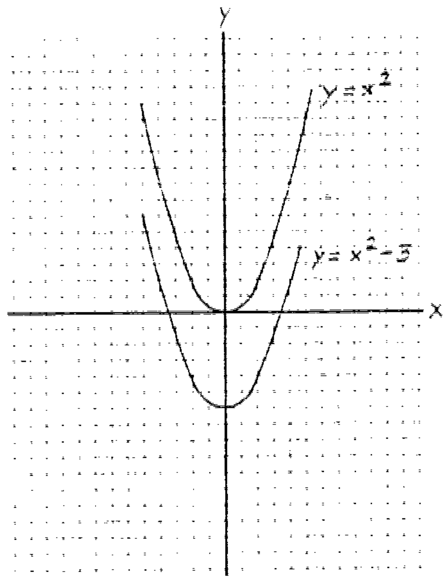
23-1. Graphs of Equations of the Form $y = ax^2 + k$

7.

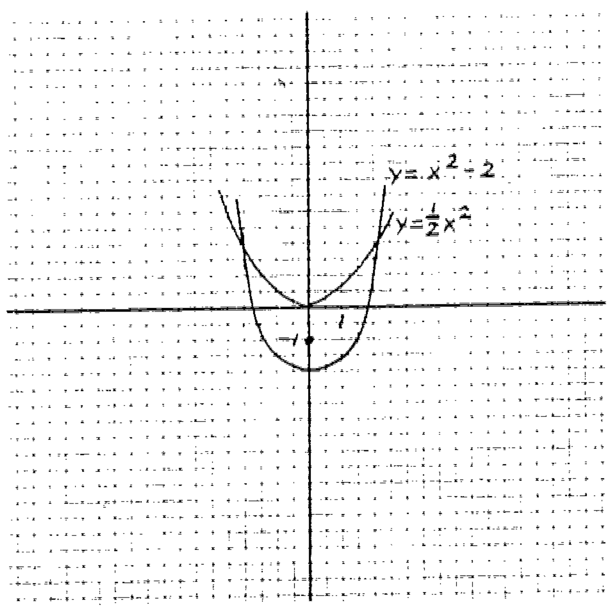


591

[illegible].

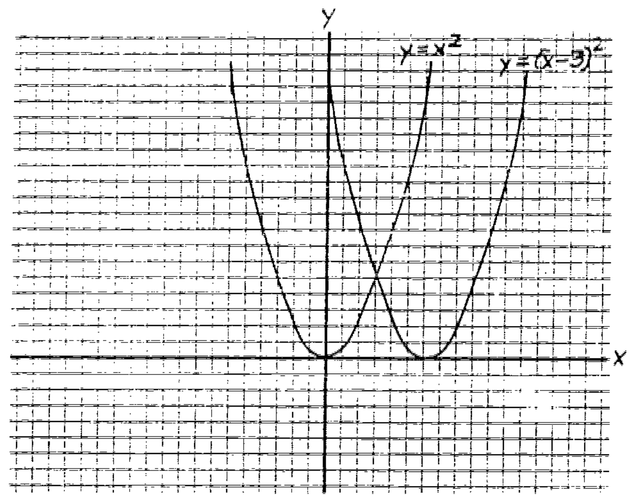


66, 67.

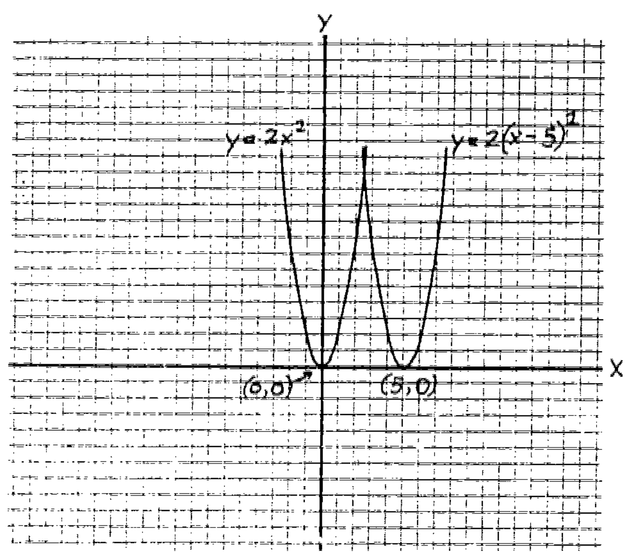


23-2. Graphs of Equations of the Form $y = a(x - h)^2 + k$

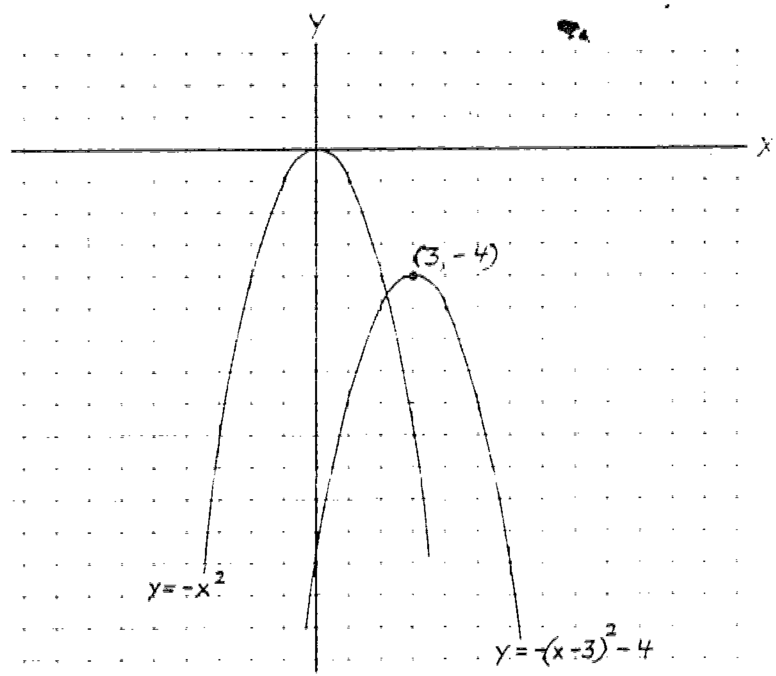
2.



16.



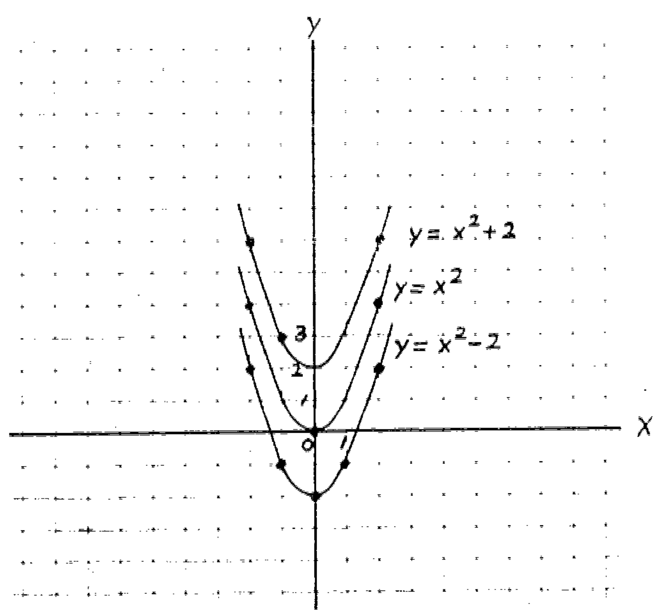
51.



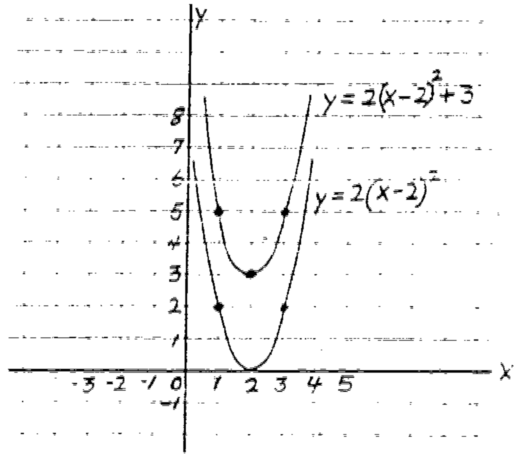
52. The graphs in parts (a), (c), (d), (e).

53. The number a.

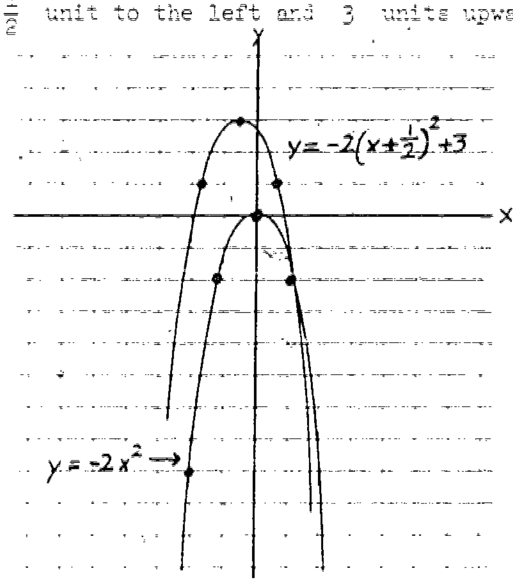
54. The graph of $y = x^2 - 2$ can be obtained by moving the graph of $y = x^2$ two units downward.
The graph of $y = x^2 + 2$ can be obtained by moving the graph of $y = x^2$ two units upward.



55. The graph of $y = 2(x - 2)^2 + 3$ can be obtained by moving the graph of $y = 2(x - 2)^2$ three units upward.



56. The graph of $y = -2(x + \frac{1}{2})^2 + 3$ can be obtained by moving the graph of $y = -2x^2$ $\frac{1}{2}$ unit to the left and 3 units upward.



57. (a) The graph of $y = 3(x - 7)^2 + \frac{1}{2}$ can be obtained by moving the graph of $y = 3x^2$ seven units to the right and $\frac{1}{2}$ unit upward.

(b) The graph of $y = 3(x - \frac{1}{2})^2 + 7$ can be obtained by moving the graph of $y = 3x^2$ $\frac{1}{2}$ unit to the right and 7 units upward.

(c) The graph of $y=2x^2 + \frac{5}{2}$ can be obtained by moving the graph of $y = 2x^2$ $\frac{5}{2}$ units upward.

(d) The graph of $y = 2(x + \frac{5}{2})^2$ can be obtained by moving the graph of $y = 2x^2$ $\frac{5}{2}$ units to the left.

(e) The graph of $y = -(x + 3)^2 - 4$ can be obtained by moving the graph of $y = -x^2$ 3 units to the left and 4 units downward.

(f) The graph of $y = x^2 + 14$ can be obtained by moving the graph of $y = x^2$ 14 units upward.

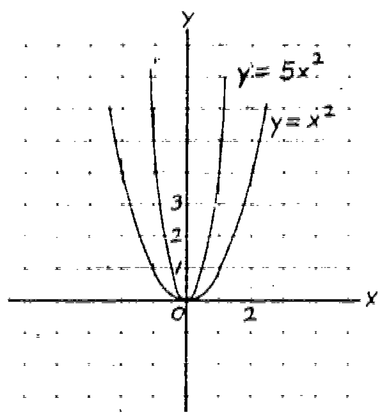
(g) The graph of $y = [illegible]x^2 + [illegible]$ can be obtained by moving the graph of $y = [illegible]x^2$ [illegible] units upward.

(h) The graph of $y = 5(x - 2)^2 + \frac{1}{2}$ can be obtained by moving the graph of $y = 5x^2$ 2 units to the right and $\frac{1}{2}$ units upward.

(i) The graph of $y = -3(x - 3)^2 - 6$ can be obtained by moving the graph of $y = -3x^2$ 3 units to the right and 6 units downward.

(j) The graph of $y = 4(3 - x)^2 - 6$ can be obtained by moving the graph of $y = 4x^2$ 3 units to the right and 6 units downward. $(3 - x)^2 = (x - 3)^2$ is true for all values of x.

58.



59. (a) Coordinates of vertex: (0, 0)
Equation of axis: $x = 0$

(b) Coordinates of vertex: (0, 0)
Equation of axis: $x = 0$

(c) Coordinates of vertex: (0, 0)
Equation of axis: $x = 0$

(d) Coordinates of vertex: (2, 0)
Equation of axis: $x = 2$

(e) Coordinates of vertex: (2, 3)
Equation of axis: $x = 2$

(f) Coordinates of vertex: (2, -3)
Equation of axis: $x = 2$

(g) Coordinates of vertex: (-2, 0)
Equation of axis: $x = -2$

(h) Coordinates of vertex: $(-2, \frac{1}{2})$
Equation of axis: $x = -2$

(i) Coordinates of vertex: $(-2, -\frac{1}{2})$
Equation of axis: $x = -2$

23-4. Summary and Review

19. If ℓ is the length, then $47 - \ell$ is the width. An open sentence is:

$$\ell(47 - \ell) = 496.$$

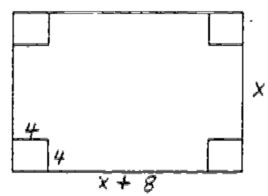
Thus,

$$-\ell^2 + 47\ell - 496 = 0$$
$$(\ell - 16)(\ell - 31) = 0.$$

The dimensions of the rectangle are 31 feet and 16 feet.

20. If x is the width in inches of the original rectangular sheet, then $x + 8$ is the length in inches. The dimensions in inches of the box are: $x - 4$, $x + 8 - 4$, 2. Hence, we have the open sentence

$$2(x - 4)(x + 4) = 256.$$



Thus,

$$x^2 - 144 = 0$$
$$(x - 12)(x + 12) = 0$$

The original box was 20 inches long and 12 inches wide.

21. If x is the number, then $x + \frac{1}{x} = 4$. Noting that $x \neq 0$, since x has a reciprocal, we may write:

$$x^2 + 1 = 4x$$
$$x^2 - 4x + 1 = 0$$
$$(x - 2)^2 - 3 = 0$$
$$(x - 2 + \sqrt{3})(x - 2 - \sqrt{3}) = 0$$

The number is either $2 - \sqrt{3}$ or $2 + \sqrt{3}$.

Note that
$$\frac{1}{2 - \sqrt{3}} = \frac{1}{2 - \sqrt{3}} \cdot \frac{2 + \sqrt{3}}{2 + \sqrt{3}} = \frac{2 + \sqrt{3}}{1} = 2 + \sqrt{3}.$$

That is, $2 + \sqrt{3}$ and $2 - \sqrt{3}$ are reciprocals.

22. Standard form: $y = 3(x - \frac{3}{2})^2 - \frac{27}{4}$.

Vertex: $(\frac{3}{2}, -\frac{27}{4})$

Points of intersection with x-axis: $(3, 0)$, $(0, 0)$.

23. Standard form: $y = (x + 1)^2 - 9$

Vertex: $(-1, -9)$

Points of intersection with x-axis: $(-4, 0)$, $(2, 0)$.

24. Standard form: $y = x^2 + 6$

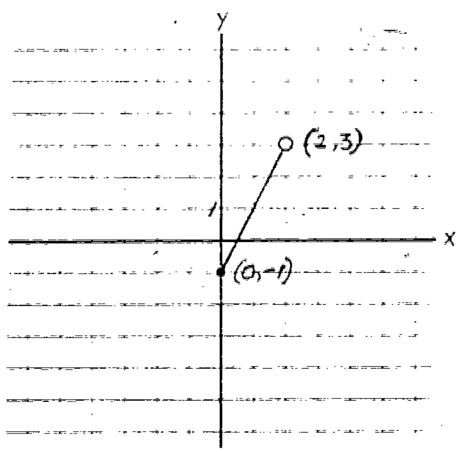
Vertex: $(0, 6)$

Points of intersection with x-axis: None.

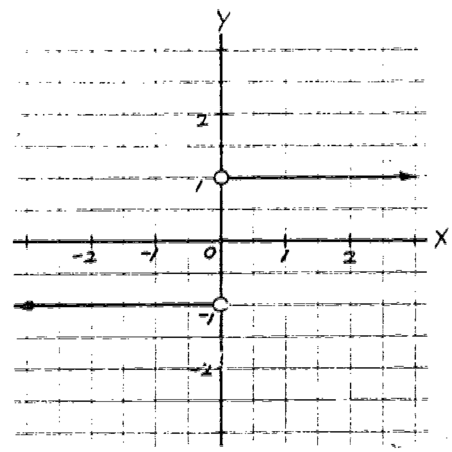
Chapter 24 - FUNCTIONS

24-3. Graphs of Functions

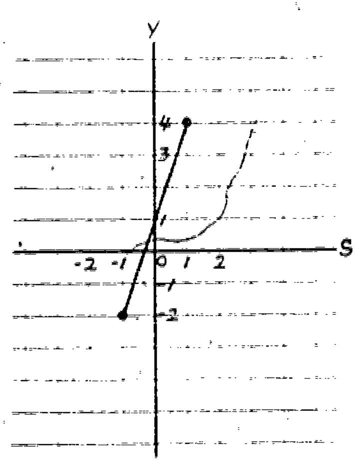
3.



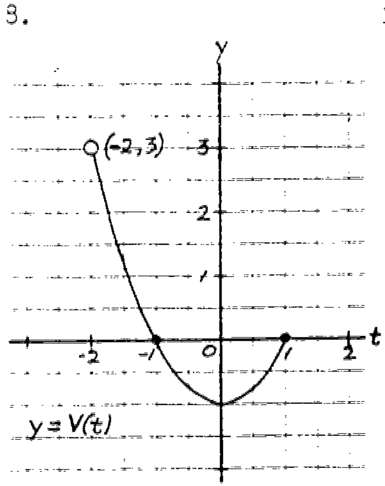
5.



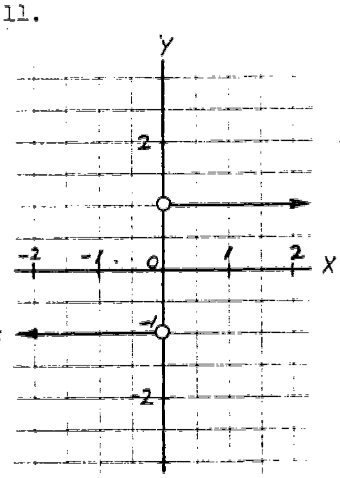
6.



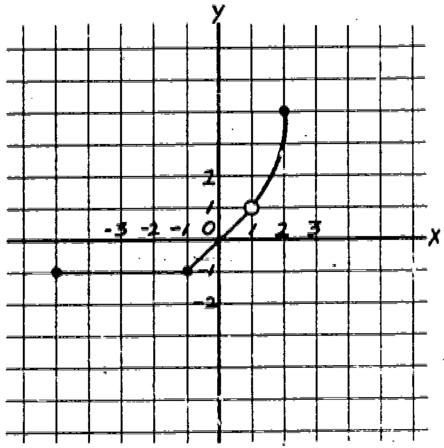
8.



11.

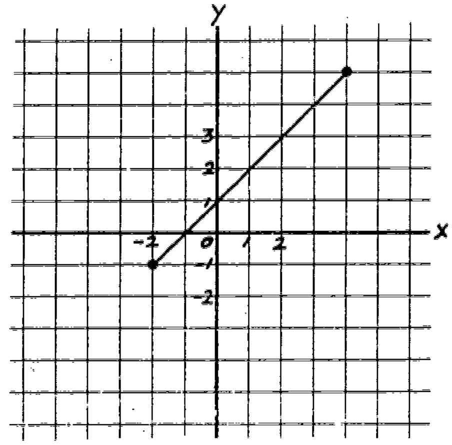


16.



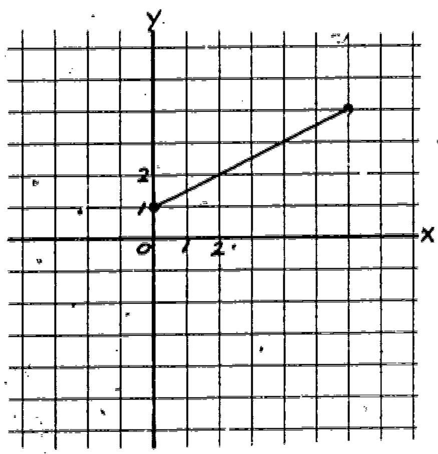
Y
X
-3 -2 -1 0 1 2 3
2
1
-1
-2

17.



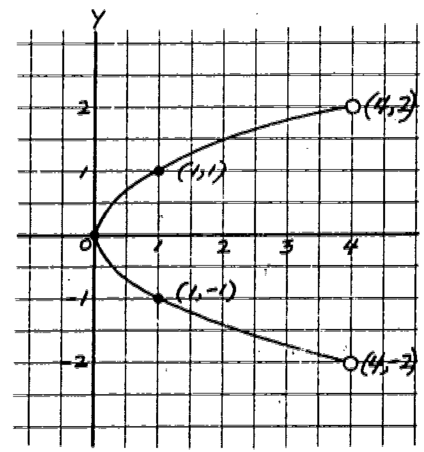
Y
X
3
2
1
-2 0 1 2
-1
-2

24.



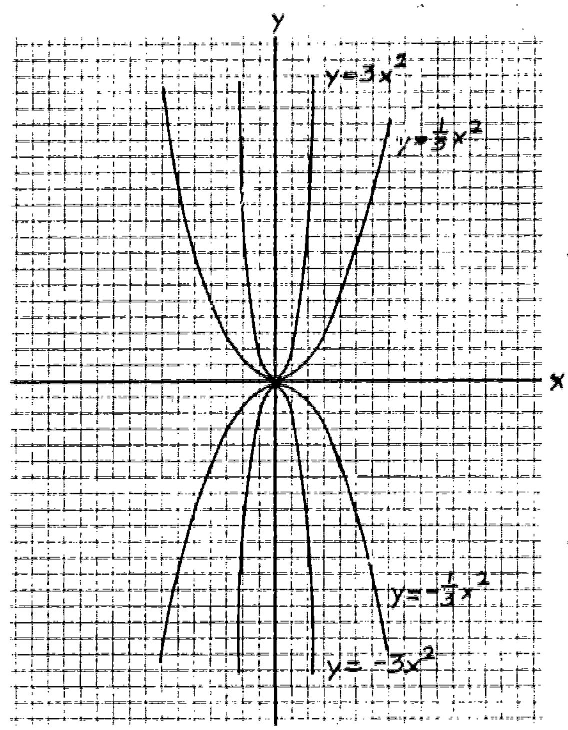
Y
X
2
1
0 1 2

44.

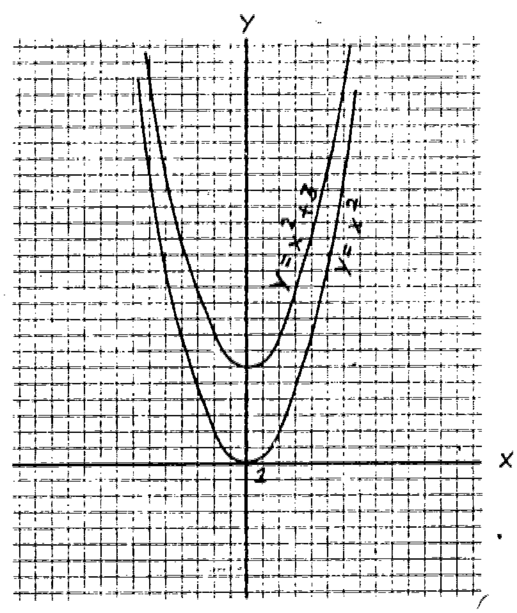


Y
X
(4,2)
(1,1)
0 1 2 3 4
(1,-1)
(4,-2)
2
1
-1
-2

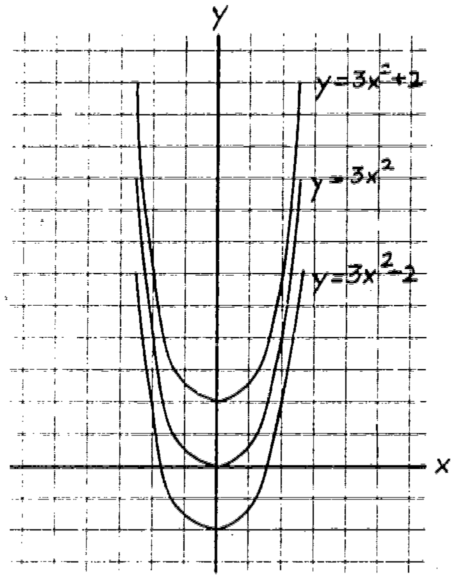
20.



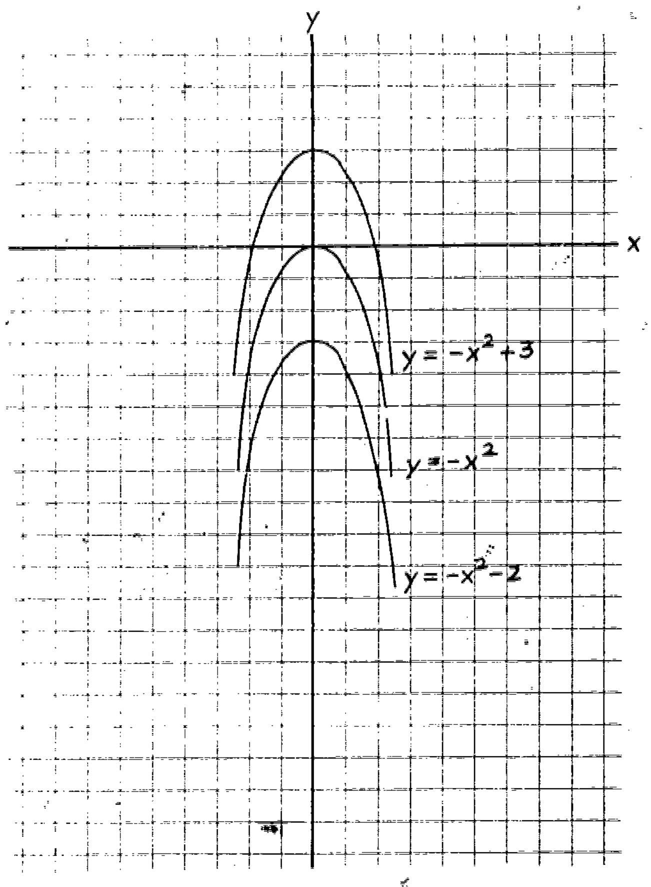
44.



68., 69., 70.



70., 71., 72.



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SYMBOLS

$\{\ \}$	indicating a set
$\emptyset$	the empty set, the null set
$=$	equals, names the same number as
$\neq$	does not equal, is different from
$>$	is greater than
$<$	is less than
$\geq$	is greater than or equal to
$\leq$	is less than or equal to
$\ngtr$	is not greater than
$\nless$	is not less than
$\cup$	union
$\cap$	intersection
$\ldots$	and so forth
$(\ )$	parentheses
$\approx$	is approximately equal to
$\lvert\ \rvert$	absolute value
$\sqrt{\ }$	non-negative square root
$\sqrt[3]{\ }$	cube root

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